Four-Dimensional Gravity on a Covariant Noncommutative Space and Unification with Internal Interactions

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Preliminaries

Second order formulation (Einstein gravity):

- metric tensor $g_{\mu\nu}$
- curvature parametrized by Riemann tensor: $R^{\rho}_{\mu\nu\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$
- torsion: $T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0$
- Christoffel Symbols: $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$
- action: $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \rightarrow$ Einstein Field Equations

First order formulation:

- vierbein and spin connection ${e_\mu}^a, {\omega_\mu}^{ab}$
- curvature parametrized by the curvature 2-form: $R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} - \omega_{\mu ac}{\omega_\nu}^c{}_b - \omega_{\nu ac}{\omega_\mu}^c{}_b$
- torsion: ${T_{\mu\nu}}^a = \partial_\mu {e_\nu}^a - \partial_\nu {e_\mu}^a + {\omega_\mu}^a{}_b {e_\nu}^c - {\omega_\nu}^a{}_b {e_\mu}^c$
- action: $S = \frac{1}{16\pi G}\int \frac{1}{2}\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}$ (Palatini action)
- $\rightarrow$ Einstein Field Equations + Torsionless condition

Approach of Einstein 4d gravity as a gauge theory

The algebra

- Employ the first order formulation of GR
- Gauge theory of Poincaré group ISO(1,3)
- Ten generators (Translations P_a & LT M_{ab})

see for details:
Utiyama '56, Kibble '61,
McDowell-Mansuri '77,
Chamseddine-West '77,
Ivanov-Niederle '82,
Kibble-Stelle '85,
Wilczek '98, Ortin '04

Generators satisfy the commutation relations:

$$[M_{ab}, M_{cd}] = \eta_{ac} M_{db} - \eta_{bc} M_{da} - \eta_{ad} M_{cb} + \eta_{bd} M_{ca}$$
$$[P_a, M_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b\,, \qquad [P_a, P_b] = 0$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ and $a, b, c, d = 1, \ldots, 4$.

The gauging procedure

- Introduction of a gauge vector field for each generator: 6 fields ${\omega_\mu}^{ab}$ for the local $SO(1,3)$ (LT)
- 4 quantities that:
  - spacetime vector field (lives on $T_x M$)
  - vector extension of $SO(1,3)$
    → appropriate choice: the 4 ${e_\mu}^a$ (invertible)
- The gauge connection is:

$$A_\mu(x) = {e_\mu}^a(x) P_a + \tfrac{1}{2}{\omega_\mu}^{ab}(x) M_{ab}$$

- Transforms in the adjoint rep, according to the rule:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

- The gauge transformation parameter, $\epsilon(x)$ is expanded as:

$$\epsilon(x) = \xi^a(x) P_a + \tfrac{1}{2}\lambda^{ab}(x) M_{ab}$$

- *Combining* the above → transformations of the fields:

$$\delta {e_\mu}^a = \partial_\mu \xi^a - {e_\mu}^b {\lambda^a}_b + {\omega_\mu}^{ab} \xi_b$$
$$\delta {\omega_\mu}^{ab} = \partial_\mu \lambda^{ab} - {\lambda^a}_c {\omega_\mu}^{cb} + {\lambda^b}_c {\omega_\mu}^{ca}$$

- Gauge transformations ↔ diffeomorphism transformations

Curvature and Torsion

- Curvatures of the fields are given by:

$$R_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Tensor $R_{\mu\nu}$ is also valued in Poincaré algebra:

$$R_{\mu\nu}(A) = T_{\mu\nu}{}^{a} P_a + \tfrac{1}{2} R_{\mu\nu}{}^{ab} M_{ab}$$

- *Combining* the above → component tensor curvatures:

$$T_{\mu\nu}{}^{a} = \partial_\mu e_\nu{}^{a} - \partial_\nu e_\mu{}^{a} + e_\mu{}^{b} \omega_{\nu b}{}^{a} - e_\nu{}^{b} \omega_{\mu b}{}^{a}$$
$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{cb} \omega_\nu{}^{a}{}_{c} + \omega_\mu{}^{ac} \omega_{\nu c}{}^{b}$$

- Palatini action is considered
- Torsionless condition + Field equations

Gauge theory of SO(2,3)

- Instead of the Poincaré group - Anti-de Sitter group: SO(2,3)
- Same amount of generators BUT they can be written on equal footing (semisimple group):

$$[\hat{M}_{AB}, \hat{M}_{CD}] = \eta_{AC}\hat{M}_{DB} - \eta_{BC}\hat{M}_{DA} - \eta_{AD}\hat{M}_{CB} + \eta_{BD}\hat{M}_{CA}$$

- η_{AB} is the 5-dim Minkowski metric with two timelike coefficients (1st and 5th) and $A, \ldots, D = 1 \ldots 5$
- Perform a splitting of the indices $A = (a, 5)$
- Define $\hat{M}_{ab} = M_{ab}$ and $\hat{M}_{a5} = \frac{1}{m}P_a$, $[m] = L^{-1}$
- Gauge connection: $A_\mu = \frac{1}{2}\hat{\omega}_\mu{}^{AB}\hat{M}_{AB} = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a$
- where $\hat{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab}$ and $\hat{\omega}_\mu{}^{a5} = m e_\mu{}^a$
- The same for the field strength tensor $\hat{R}_{\mu\nu}{}^{AB}$:

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + 2m^2 e_\mu{}^{[a} e_\mu{}^{b]}\,, \quad \hat{R}_{\mu\nu}{}^{a5} = m T_{\mu\nu}{}^a$$

- Consider the following $SO(2,3)$ invariant quadratic action:

$$S = a_{AdS} \int d^4x \Big(m y^E \epsilon_{ABCDE} \frac{1}{4} \hat{R}_{\mu\nu}{}^{AB} \hat{R}_{\rho\sigma}{}^{CD} \epsilon^{\mu\nu\rho\sigma} + \\ + \lambda \left(y^E y_E + m^{-2} \right) \Big)$$

- y^E an internal space vector field
- vector taken to be gauge fixed towards the 5-th direction:

$$y = y^0 = \left(0,0,0,0,m^{-1}\right).$$

- the non-vanishing value $y^5(x)$ is responsible for the symmetry breaking of $SO(2,3)$ to the $SO(1,3)$

$$S = \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu}{}^{ab} \hat{R}_{\rho\sigma}{}^{cd} \epsilon_{abcd}$$
$$= \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left(\mathcal{L}_{RR} + m^2 \mathcal{L}_{eeR} + m^4 \mathcal{L}_{eeee} \right)$$

- $\mathcal{L}_{RR}$: Gauss-Bonnet - no contribution to the e.o.m.
- $\mathcal{L}_{eeR}$: Palatini action (torsionless + Einstein Field Equations)
- $\mathcal{L}_{eeee}$: Plays the role of cosmological constant
- Solution of Einstein Field Equations is the Anti-de Sitter space
- If $m \to 0$: Minkowski spacetime (flat solution).

Conformal 4d gravity as a gauge theory

- Group parametrizing the symmetry: $SO(2,4)$
- 15 generators: 6 LT M_{ab}, 4 translations, P_a, 4 conformal boosts K_a and the dilatation D
- Following the same procedure one calculates transf of the gauge fields and tensors after defining the gauge connection
- Action is taken of $SO(2,4)$ invariant quadratic form
- Initial symmetry breaks under certain constraints resulting to the *Weyl action* *Kaku, Townsend, Van Nieu/zen '77, Fradkin, Tseytlin '85*
- Initial symmetry breaks spontaneously by introducing a scalar in the adjoint rep fixed in the dilatation direction, or by two scalars in vector reps.

Roumelioti, Stefas, Z '24

SSB by using a scalar in the adjoint representation

Gauge connection:

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

Field strength tensor:

$$F_{\mu\nu} = \frac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + \tilde{R}_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a K_a + R_{\mu\nu} D,$$

where

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b - 8e_{[\mu}{}^{[a}b_{\nu]}{}^{b]} \\
&= R^{(0)ab}_{\mu\nu} - 8e_{[\mu}{}^a b_{\nu]}{}^{b]}, \\
\tilde{R}_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\nu{}^{ab}e_{\mu b} - 2\tilde{a}_{[\mu}e_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu} - 2\tilde{a}_{[\mu}e_{\nu]}{}^a, \\
R_{\mu\nu}{}^a &= \partial_\mu b_\nu{}^a - \partial_\nu b_\mu{}^a + \omega_\mu{}^{ab}b_{\nu b} - \omega_\nu{}^{ab}b_{\mu b} + 2\tilde{a}_{[\mu}b_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(b) + 2\tilde{a}_{[\mu}b_{\nu]}{}^a, \\
R_{\mu\nu} &= \partial_\mu\tilde{a}_\nu - \partial_\nu\tilde{a}_\mu + 4e_{[\mu}{}^a b_{\nu]a},
\end{aligned}$$

We start with the parity conserving action, which is quadratic in terms of the field strength tensor and introduce a scalar in the rep 15

$$S_{SO(2,4)} = a_{CG} \int d^4x \left[\mathrm{tr}\, \epsilon^{\mu\nu\rho\sigma} m\phi F_{\mu\nu} F_{\rho\sigma} + \left(\phi^2 - m^{-2} \mathbb{1}_4\right) \right],$$

The scalar expanded on the generators is:

$$\phi = \phi^{ab} M_{ab} + \tilde{\phi}^a P_a + \phi^a K_a + \tilde{\phi} D,$$

We pick the specific gauge in which ϕ is diagonal of the form $diag(1,1,-1,-1)$. Specifically we choose ϕ to be only in the direction of the dilatation generator D:

$$\phi = \phi^0 = \tilde{\phi} D \xrightarrow{\phi^2 = m^{-2} \mathbb{1}_4} \phi = -2m^{-1} D.$$

The resulting broken action is (after employing anticommutator relations and the traces over the generators):

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd}$$

The $\tilde{a}_\mu$ is not present in the action, so we can set it equal to zero.

$R_{\mu\nu}$ is also absent so we can also set it equal to zero

$$R_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + 4 e_{[\mu}{}^a b_{\nu]a} = 0 \xrightarrow{\tilde{a}_\mu = 0}$$

$$e_\mu{}^a b_{\nu a} - e_\nu{}^a b_{\mu a} = 0$$

We examine two possible solutions of the above equation:

- $b_\mu{}^a = a e_\mu{}^a$, *Chamseddine '03*
- $b_\mu{}^a = -\frac{1}{4}\left(R_\mu{}^a + \frac{1}{6} R e_\mu{}^a\right)$ *Kaku, Townsend, van Nieuwenhuizen, 78*

Freedman, Van Proyen "Supergravity" '12

The first choice leads to the Einstein-Hilbert action, while the second leads to Weyl action.

Einstein-Hilbert action

- When $b_\mu{}^a = a e_\mu{}^a$, the broken action becomes:

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} \implies$$

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \Big[R^{(0)ab}_{\mu\nu} R^{(0)cd}_{\rho\sigma} - 16 m^2 a R^{(0)ab}_{\mu\nu} e_\rho{}^c e_\sigma{}^d + \\ + 64 m^4 a^2 e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d \Big]$$

This action consists of three terms: one G-B topological term, the E-H action, and a cosmological constant. For $a < 0$ describes GR in AdS space.

Weyl action

- When $b_\mu{}^a = -\frac{1}{4}(R_\mu{}^a + \frac{1}{6}Re_\mu{}^a)$, the broken action becomes

$$S = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu} - \frac{1}{2}\left(\tilde{e}_\mu{}^{[a}R_\nu{}^{b]} - \tilde{e}_\nu{}^{[a}R_\mu{}^{b]}\right) + \\ + \frac{1}{3}R\tilde{e}_\mu{}^{[a}\tilde{e}_\nu{}^{b]}\Big] \\ \Big[R^{(0)cd}_{\rho\sigma} - \frac{1}{2}\left(\tilde{e}_\rho{}^{[c}R_\sigma{}^{d]} - \tilde{e}_\sigma{}^{[c}R_\rho{}^{d]}\right) + \\ + \frac{1}{3}R\tilde{e}_\rho{}^{[c}\tilde{e}_\sigma{}^{d]}\Big],$$

where $\tilde{e}_\mu{}^a = me_\mu{}^a$ is the rescaled vierbein. The above action is equal to

$$S = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}C_{\mu\nu}{}^{ab}C_{\rho\sigma}{}^{cd},$$

where $C_{\mu\nu}{}^{ab}$ is the Weyl conformal tensor.

The nc framework & gauge theories

- Quantization of phase space of $x^i, p_j \rightarrow$ replace with Herm operators: $\hat{x}^i, \hat{p}_j$ satisfying:$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j$
- Noncommutative space $\rightarrow$ quantization of space: $x^i \rightarrow$ replace with operators $\hat{x}^i$ ($\in \mathcal{A}$) satisfying: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x})$

Connes '94, Madore '99

- Antisymmetric tensor $\theta^{ij}(\hat{x})$ - defines the nc of the space
  - Canonical case: $\theta^{ij}(\hat{x}) = \theta^{ij}, i, j = 1, \ldots, N$
    For $N = 2 \rightarrow$ *Moyal plane*
  - Lie-type case: $\theta^{ij}(\hat{x}) = C^{ij}{}_k \hat{x}^k, i, j = 1, \ldots, N$
    For $N = 3 \rightarrow$ Noncommutative *(fuzzy) sphere* (SU(2))
- nc framework admits a matrix representation (operators)
  - Derivation: $e_i(A) = [d_i, A]$, $d_i \in \mathcal{A}$
  - Integration $\rightarrow$ Trace

For Reviews:
Szabo '01, Douglas-Nekrasov '01

The nc gauge theories

- Natural intro of nc gauge theories through *covariant nc coordinates*: $\mathcal{X}_\mu = X_\mu + A_\mu$ *Madore-Schraml-Schupp-Wess '00*
- Obeys a covariant gauge transformation rule: $\delta \mathcal{X}_\mu = i[\epsilon, \mathcal{X}_\mu]$
- A_μ transforms in analogy with the gauge connection: $\delta A_\mu = -i[X_\mu, \epsilon] + i[\epsilon, A_\mu]$, ($\epsilon$ - the gauge parameter)
- Definition of a nc *covariant field strength tensor* depends on the space:
  - Canonical case: $F_{ab} = [\mathcal{X}_a, \mathcal{X}_b] - i\theta_{ab}$
  - Lie-Type case: $F_{ab} = [\mathcal{X}_a, \mathcal{X}_b] - iC_{abc}\mathcal{X}_c$

Non-Abelian case

▷ *In nonabelian case, where are the gauge fields valued?*

- Let us consider the CR of two elements of an algebra:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\}$$

- *Not* possible to restrict to a matrix algebra:
  last term neither *vanishes* in nc nor is an *algebra element*

- There are two options to overpass the difficulty:

  Ćirić-Gočanin-Konjik-Radovanović '18

  - Consider the universal enveloping algebra
  - Extend the generators and/or fix the rep so that the anticommutators close

▷ *We employ the second option*

The 4d covariant noncommutative space

Motivation for a 4d covariant nc space

- Constructing field theories on nc spaces is non-trivial: nc deformations break Lorentz invariance
- such an example is the fuzzy sphere (2d space) - coords are identified as rescaled SU(2) generators

Madore '92
Hammou-Lagraa-Sheikh Jabbari '02
Vitale-Wallet '13, Vitale '14
Jurman-Steinacker '14
Chatzistavrakidis-Jonke-Jurman-Manolakos-Manousselis-GZ '18

- Previous work on 3d nc gravity on the covariant spaces $R^3_\lambda (R^{1,2}_\lambda)$
- Need of 4d covariant nc space to construct a gravity gauge theory

Construction of the 4d covariant nc space

- dS_4: homogeneous spacetime with constant curvature (positive)
- Described by the embedding $\eta^{AB} X_A X_B = R^2$ into M_5
- Aim for a nc version of dS_4

Snyder '47

- The SO(1,4) generators, $J_{mn}, m, n = 0, \ldots, 4$, satisfy the commutation relation:

$$[J_{mn}, J_{rs}] = i(\eta_{mr}J_{ns} + \eta_{ns}J_{mr} - \eta_{nr}J_{ms} - \eta_{ms}J_{nr})$$

- Consider decomposition of $SO(1,4)$ to maximal subgroup, $SO(1,3)$
- Introduce a length parameter λ and define operators as rescalings of the generators
- Thus, the commutation relations regarding the operators $\Theta_{\mu\nu}$ and X_μ are:

$$[\Theta_{ij}, \Theta_{kl}] = i\hbar\left(\eta_{ik}\Theta_{jl} + \eta_{jl}\Theta_{ik} - \eta_{jk}\Theta_{il} - \eta_{il}\Theta_{jk}\right),$$
$$[\Theta_{ij}, X_k] = i\hbar\left(\eta_{ik}X_j - \eta_{jk}X_i\right),$$
$$[X_i, X_j] = \frac{i\lambda^2}{\hbar}\Theta_{ij}$$

- The noncommutativity of coordinates becomes manifest

Yang's Model '47

- Requiring covariance → use a group with larger symmetry → minimum extension: SO(1,5)

Yang '47
Kimura '02, Heckman-Verlinde '15
Steinacker '16
Sperling-Steinacker '17,'19
Burić-Madore '14,'15
Manousselis-Manolakos-GZ '19,'21

- The SO(1,5) generators, $J_{MN}, M, N = 0, \ldots, 5$, satisfy the commutation relation:

$$[J_{MN}, J_{P\Sigma}] = i(\eta_{MP} J_{N\Sigma} + \eta_{N\Sigma} J_{MP} - \eta_{NP} J_{M\Sigma} - \eta_{M\Sigma} J_{NP})$$

- Employ a *2-step* decomposition $SO(1,5) \supset SO(1,4) \supset SO(1,3)$

Yang's Model '47 (Continued)

- Introduce a length parameter λ and define operators as rescalings of the generators (like in Snyder's case)
- Thus, the commutation relations regarding all the operators $\Theta_{\mu\nu}, X_\mu, P_\mu, h$ are:

$$
\begin{aligned}
[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] &= i\hbar(\eta_{\mu\rho}\Theta_{\nu\sigma} + \eta_{\nu\sigma}\Theta_{\mu\rho} - \eta_{\nu\rho}\Theta_{\mu\sigma} - \eta_{\mu\sigma}\Theta_{\nu\rho})\,, \\
[\Theta_{\mu\nu}, X_\rho] &= i\hbar(\eta_{\mu\rho}X_\nu - \eta_{\nu\rho}X_\mu) \\
[\Theta_{\mu\nu}, P_\rho] &= i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[P_\mu, P_\nu] &= i\frac{\hbar}{\lambda^2}\Theta_{\mu\nu}\,, \qquad [X_\mu, X_\nu] = i\frac{\lambda^2}{\hbar}\Theta_{\mu\nu}\,, \\
[P_\mu, h] &= -i\frac{\hbar}{\lambda^2}X_\mu\,, \qquad [X_\mu, h] = i\frac{\lambda^2}{\hbar}P_\mu\,, \\
[P_\mu, X_\nu] &= i\hbar\eta_{\mu\nu}h\,, \qquad [\Theta_{\mu\nu}, h] = 0
\end{aligned}
$$

- The above relations describe the noncommutative space

Noncommutative gauge theory of 4d gravity

- Formulation of gravity on the above space
- Noncommutative gauge theory construction + the procedure described in the Einstein gravity case

Kimura '02, Heckman-Verlinde '15

- Gauge the isometry group of the space, $\mathrm{SO}(1,4)$ as seen as a subgroup of the $\mathrm{SO}(1,5)$ we ended up
- Anticommutators do not close $\rightarrow$ enlargement of the algebra + fix the representation

Aschieri-Castellani '09

Chatzistavrakidis-Jonke-Jurman-Manolakos-Manousselis-GZ '18

- Noncommutative gauge theory of $\mathrm{SO}(2,4) \times \mathrm{U}(1)$
- The generators of the group are represented by combinations of the 4x4 gamma matrices

- Specifically, the generators are expressed by:
  - six Lorentz rotation generators: $M_{ab} = -\frac{i}{4}\left[\gamma_a, \gamma_b\right]$
  - four generators for conformal boosts: $K_a = \frac{1}{2}\gamma_a(1+\gamma_5)$
  - four generators for translations: $P_a = -\frac{1}{2}\gamma_a(1-\gamma_5)$
  - one generator for special conformal transformations: $\mathrm{D} = -\frac{1}{2}\gamma_5$
  - one $\mathrm{U}(1)$ generator: $\mathbb{1}$
- The above expressions of the generators allow the calculation of the algebra they satisfy:

$$\begin{aligned}
&[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac},\\
&[K_a, P_b] = -2\left(\eta_{ab}D + M_{ab}\right),\ [P_a, D] = P_a,\ [K_a, D] = -K_a,\\
&[M_{ab}, K_c] = \eta_{bc}K_a - \eta_{ac}K_b,\ [M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b
\end{aligned}$$

- Generators satisfy the following anticommutation relations:

Smolin '03

$$
\begin{aligned}
\{M_{ab}, M_{cd}\} &= \frac{1}{2}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) - i\epsilon_{abcd}D, \\
\{M_{ab}, P_c\} &= +i\epsilon_{abcd}P^d, \\
\{M_{ab}, K_c\} &= -i\epsilon_{abcd}K^d, \\
\{M_{ab}, D\} &= 2M_{ab}D, \\
\{P_a, K_b\} &= 4M_{ab}D + \eta_{ab}, \\
\{K_a, K_b\} &= \{P_a, P_b\} = -\eta_{ab}, \\
\{P_a, D\} &= \{K_a, D\} = 0.
\end{aligned}
$$

- We will introduce gauge fields in a motivated way
- Use the general treatment of nc gauge theories

NC gauge theory and the action

Manolakos, Manousselis, GZ '21

- Start with the following action:

$$\mathcal{S} = \mathrm{Tr}\left([X_\mu, X_\nu] - \kappa^2\Theta_{\mu\nu}\right)\left([X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}$$

- Field equations satisfied by the nc space for $\kappa^2 = i\lambda^2/\hbar$
- Introduce gauge fields as fluctuations:

$$\mathcal{S} = \mathrm{Trtr}\epsilon^{\mu\nu\rho\sigma}\left([X_\mu + A_\mu, X_\nu + A_\nu] - \kappa^2(\Theta_{\mu\nu} + \mathcal{B}_{\mu\nu})\right)$$
$$\left([X_\rho + A_\rho, X_\sigma + A_\sigma] - \kappa^2(\Theta_{\rho\sigma} + \mathcal{B}_{\rho\sigma})\right)$$

- The above action is written:

$$\mathcal{S} = \mathrm{Trtr}\left([\mathcal{X}_\mu, \mathcal{X}_\nu] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}\right)\left([\mathcal{X}_\rho, \mathcal{X}_\sigma] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}$$

$$:= \mathrm{Trtr}\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \xrightarrow[\mathcal{X},\hat{\Theta}]{\delta\mathcal{S}} \boxed{\epsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\rho\sigma} = 0\,, \quad \epsilon^{\mu\nu\rho\sigma}[\mathcal{X}_\nu, \mathcal{R}_{\rho\sigma}] = 0}$$

- where we have defined:
  - $\mathcal{X}_\mu = X_\mu + A_\mu$, the covariant coordinate
  - $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, the covariant noncommutative tensor
  - $\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\frac{\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}$, the field strength tensor

Gauge connection and field strength tensor decompose as:

$$A_\mu(X) = e_\mu{}^a \otimes P_a + \omega_\mu{}^{ab} \otimes M_{ab} + b_\mu{}^a \otimes K_a + \tilde{a}_\mu \otimes D + a_\mu \otimes \mathbf{I}_4 \,.$$

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}{}^a \otimes P_a + R_{\mu\nu}{}^{ab} \otimes M_{ab} + R_{\mu\nu}{}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbf{I}_4 \,.$$

The component curvatures:

$$R_{\mu\nu} = [X_\mu, a_\nu] - [X_\nu, a_\mu] + [a_\mu, a_\nu] + [b_\mu{}^a, b_{\nu a}] + [\tilde{a}_\mu, \tilde{a}_\nu] + \frac{1}{2}[\omega_\mu{}^{ab}, \omega_{\nu ab}]$$
$$+ [e_{\mu a}, e_\nu{}^a] - \frac{i\hbar}{\lambda^2} B_{\mu\nu}$$

$$\tilde{R}_{\mu\nu} = [X_\mu, \tilde{a}_\nu] + [a_\mu, \tilde{a}_\nu] - [X_\nu, \tilde{a}_\mu] - [a_\nu, \tilde{a}_\mu] - i\{b_{\mu a}, e_\nu{}^a\} + i\{b_{\nu a}, e_\mu{}^a\}$$
$$+ \frac{1}{2}\epsilon_{abcd}[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}] - \frac{i\hbar}{\lambda^2} \tilde{B}_{\mu\nu}$$

$$R_{\mu\nu}{}^a = [X_\mu, b_\nu{}^a] + [a_\mu, b_\nu{}^a] - [X_\nu, b_\mu{}^a] - [a_\nu, b_\mu{}^a] + i\{b_{\mu b}, \omega_\mu{}^{ab}\} - i\{b_{\nu b}, \omega_\mu{}^{ab}\}$$
$$+ i\{\tilde{a}_\mu, e_\nu{}^a\} - i\{\tilde{a}_\nu, e_\mu{}^a\} + \epsilon_{abcd}([e_\mu{}^b, \omega_\nu{}^{cd}] - [e_\nu{}^b, \omega_\mu{}^{cd}]) - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^a$$

$$\tilde{R}_{\mu\nu}{}^a = [X_\mu, e_\nu{}^a] + [a_\mu, e_\nu{}^a] - [X_\nu, e_\mu{}^a] - [a_\nu, e_\mu{}^a] + i\{b_\mu{}^a, \tilde{a}_\nu\} - i\{b_\nu{}^a, \tilde{a}_\mu\}$$
$$- ([b_\mu{}^b, \omega_\nu{}^{cd}] - [b_\nu{}^b, \omega_\mu{}^{cd}])\epsilon_{abcd} - i\{\omega_\mu{}^{ab}, e_{\nu b}\} + i\{\omega_\nu{}^{ab}, e_{\mu b}\} - \frac{i\hbar}{\lambda^2} \tilde{B}_{\mu\nu}{}^a$$

$$R_{\mu\nu}{}^{ab} = [X_\mu, \omega_n{}^{ab}] + [a_\mu, \omega_\nu{}^{ab}] - [X_\nu, \omega_\mu{}^{ab}] - [a_\nu, \omega_m{}^{ab}] + 2i\{b_\mu{}^a, b_\nu{}^b\} + ([b_\mu{}^c, e_\nu{}^d]$$
$$- [b_\nu{}^c, e_\mu{}^d])\epsilon_{abcd} + \frac{1}{2}([\tilde{a}_\mu, \omega_\nu{}^{cd}] - [\tilde{a}_\nu, \omega_\mu{}^{cd}])\epsilon_{abcd} + 2i\{\omega_\mu{}^{ac}, \omega_\nu{}^b{}_c\}$$
$$+ 2i\{e_\mu{}^a, e_\nu{}^b\} - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^{ab}$$

Symmetry breaking

Introduction of auxiliary field $\Phi(X)$ charged under $U(1)$:

$$\Phi = \tilde{\phi}^a \otimes P_a + \phi^{ab} \otimes M_{ab} + \phi^a \otimes K_a + \phi \otimes \mathbf{I}_4 + \tilde{\phi} \otimes D$$

into the action:

$$\mathcal{S} = \mathrm{Trtr}_G \, \lambda \Phi(X) \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} + \eta(\Phi(X)^2 - \lambda^{-2} \mathbf{I}_N \otimes \mathbf{I}_4) \, ,$$

induces a symmetry breaking:

$$\boxed{\mathcal{S}_{br} = \mathrm{Tr} \left(\frac{\sqrt{2}}{4} \varepsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} - 4 R_{\mu\nu} \tilde{R}_{\rho\sigma} \right) \varepsilon^{\mu\nu\rho\sigma}}$$

when the auxiliary field is gauge fixed as:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi}=-2\lambda^{-1}} = -2\lambda^{-1} \mathbf{I}_N \otimes D$$

Residual symmetry: $SO(1,3) \times U(1)$

The constraints that correspond to the above breaking are:

Chamseddine '02

$R_{\mu\nu}{}^a = \frac{i}{2} \tilde{R}_{\mu\nu}{}^a = 0$ leading to $\tilde{a}_\mu = 0$, $b_\mu{}^a = \frac{i}{2} e_\mu{}^a$ and $B_{\mu\nu}{}^a = \frac{i}{2} \tilde{B}_{\mu\nu}{}^a$

The commutative limit

- The 2-form field, $\mathcal{B}_{\mu\nu}$ and a_μ decouple
- The commutators of functions vanish: $[f(x), g(x)] \to 0$
- The anticommutators of functions reduce to product: $\{f(x), g(x)\} \to 2f(x)g(x)$
- The inner derivation becomes: $[X_\mu, f] \to \partial_\mu f$
- Trace reduces to integration: $\frac{\sqrt{2}}{4}\mathrm{Tr} \to \int d^4x$
- We also regard the following reparametrizations:

- $e_\mu{}^a \to ime_\mu{}^a\,, \quad P_a \to -\frac{i}{m}P_a\,, \quad \tilde{R}_{\mu\nu}{}^a \to imT_{\mu\nu}{}^a$
- $\omega_\mu{}^{ab} \to -\frac{i}{2}\omega_\mu{}^{ab}\,, \quad M_{ab} \to 2iM_{ab}\,, \quad R_{\mu\nu}{}^{ab} \to -\frac{i}{2}R_{\mu\nu}{}^{ab}$

- Similar procedure to the gauge-theoretic approach of Einstein gravity is followed leading to the same results, i.e. Palatini action with cosmological constant

The transformations of the fields:

$$\delta\omega_m{}^{ab} = -i[X_m, \lambda^{ab}] - i[a_m, \lambda^{ab}] + i[\epsilon_0, \omega_m{}^{ab}] - 2\{\xi^a, b_m{}^b\} - \frac{1}{2}\{\lambda^a{}_c, \omega_m{}^{bc}\}$$
$$- \frac{1}{2}\{\tilde{\xi}^a, e_m{}^b\} + i[\xi^c, e_m{}^d]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, \omega_m{}^{cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{a}_m]\epsilon_{abcd} - i[\tilde{\xi}^c, b_m{}^d]\epsilon_{abcd}$$
$$\delta e_m{}^a = -i[X_m, \tilde{\xi}^a] - i[a_m, \tilde{\xi}^a] + i[\epsilon_0, e_m{}^a] - \{\xi^a, \tilde{a}_m\} + \{\tilde{\epsilon}_0, b_m{}^a\} + \frac{1}{4}\{\lambda^a{}_b, e_m{}^b\}$$
$$- \frac{1}{4}\{\tilde{\xi}_b, \omega_m{}^{ab}\} + i[\xi^c, \omega_m{}^{bd}]\epsilon_{abcd} - i[\lambda^{cd}, b_m{}^b]\epsilon_{abcd}$$
$$\delta b_m{}^a = -i[X_m, \xi^a] - i[a_m, \xi^a] + i[\epsilon_0, b_m{}^a] - \{\xi_b, \omega_m{}^{ab}\} - 2\{\tilde{\epsilon}_0, e_m{}^a\} + \frac{1}{2}\{\lambda^a{}_b, b_m{}^b\}$$
$$+ \{\tilde{\xi}^a, \tilde{a}_m\} + i[\lambda^{bc}, e_m{}^d]\epsilon_{abcd} + i[\tilde{\xi}^b, \omega_m{}^{cd}]\epsilon_{abcd}$$
$$\delta a_m = -i[X_m, \epsilon_0] - i[a_m, \epsilon_0] + i[\xi^a, b_m{}^a] + i[\tilde{\epsilon}_0, \tilde{a}_m] + \frac{i}{2}[\lambda_{ab}, \omega_m{}^{ab}] + \frac{i}{2}[\tilde{\xi}_a, e_m{}^a]$$
$$\delta\tilde{a}_m = -i[X_m, \tilde{\epsilon}_0] - i[a_m, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{a}_m] + \{\xi_a, e_m{}^a\} - \{\tilde{\xi}_a, b_m{}^a\} + \frac{i}{2}[\lambda^{ad}, \omega_m{}^{bc}]\epsilon_{abcd}$$

(Transformations of the component of $\mathcal{B}_{mn}$ are calculated as well)

Unification of gravity theories with Internal Interactions

- So far in the gauge theoretic approach of gravity, general relativity is described by gauging the symmetry of the tangent manifold in four dimensions.
- Usually the dimension of the tangent space is considered to be equal to the dimension of the curved manifold. However, the tangent group of a manifold of dimension d is not necessarily SO_d.

Weinberg '84

- It has been suggested that by gauging an enlarged symmetry of the tangent space in four dimensions one could unify gravity with internal interactions.

Chamseddine, Mukhanov '10

- We aim to unify gravities as a gauge theory with internal interactions under one unification gauge group.
- Attempts of unification for the case of Einstein gravity: Chamseddine and Mukhanov, 2010; Percacci, 1991; Konitopoulos, Roumelioti, GZ, 2023.

Unification group

- Weyl gravity is based on gauging the $SO(2,4)$, while Fuzzy gravity on $SO(2,4) \times U(1)$.
- Internal Interactions by $SO(10)$ (GUT).
- Spontaneous symmetry breakings are used in all cases.

Usually to have a Chiral theory we need a $SO(4n+2)$ group. The smallest unification group in which both Majorana and Weyl condition can be imposed is $SO(2,16)$ from which:

$$SO(2,16) \xrightarrow{SSB} SO(2,4) \times SO(12)$$

and

$$SO(12) \xrightarrow{SSB} SO(10) \times [U(1)].$$

Breakings and branching rules

We start from $SO(2,16) \sim SO(18)$

- For CG we gauge $SO(2,4) \sim SU(2,2) \sim SO(6) \sim SU(4)$
- For FG we gauge $SO(2,4) \times U(1) \sim SO(6) \times U(1) \sim U(4)$
- For internal interactions we require $SO(10)$ GUT.

$$C_{SO(2,16)}(SO(2,4)) = SO(10)$$

and

$$C_{SO(2,16)}(SO(2,4) \times U(1)) = SO(10) \times U(1).$$

Breakings and branching rules (Continued)

$$SO(18) \supset SU(4) \times SO(12)$$

$$\begin{aligned} 18 &= (6,1) + (1,12) && \text{vector} \\ 153 &= (15,1) + (6,12) + (1,66) && \text{adjoint} \\ 256 &= (4,\overline{32}) + (\bar{4},32) && \text{spinor} \\ 170 &= (1,1) + (6,12) + (20',1) + (1,77) && \text{2nd rank symmetric} \end{aligned}$$

VEV in the $\langle 1,1 \rangle$ component of a scalar in 170 leads to $SU(4) \times SO(12)$.

Breakings and branching rules (Continued)

We break the $SO(12)$ down to $SO(10) \times U(1)$ or to $SO(10)$ with the 66 rep or the 77 rep.

$$SO(12) \supset SO(10) \times U(1)$$
$$66 = (1)(0) + (10)(2) + (10)(-2) + (45)(0)$$
$$77 = (1)(4) + (1)(0) + (1)(-4) + (10)(2) + (10)(-2) + (54)(0)$$

by giving VEV to the $\langle(1)(0)\rangle$ of the 66 rep we obtain $SO(10) \times U(1)$.

by giving VEV to the $\langle(1)(4)\rangle$ of the 77 rep we obtain $SO(10)$.

Breakings and branching rules (Continued)

We break $SU(4)$ in 2 steps:

- First step: Breaking $SU(4) \to Sp_4$:

$$
\begin{aligned}
SU(4) &\supset Sp4 \\
4 &= 4 \\
6 &= 1 + 5
\end{aligned}
$$

  giving VEV to a scalar in 6 rep in the $\langle 1 \rangle$ component, the $SU(4)$ breaks down to the Sp_4.

- Second step: Breaking $Sp_4 \to SU(2) \times SU(2)$

$$
\begin{aligned}
Sp_4 &\supset SU(2) \times SU(2) \\
5 &= (1,1) + (2,2) \\
4 &= (2,1) + (1,2).
\end{aligned}
$$

  giving VEV in $\langle 1, 1 \rangle$ of a scalar in the 5 rep we obtain eventually the Lorentz group $SU(2) \times SU(2) \sim SO(1,3)$.

Fermions

Weyl condition: $\Gamma^{D+1}\psi_{\pm} = \pm\psi_{\pm}, \quad D = even.$

Note that since $\Gamma^{D+1} = \gamma^5 \otimes \gamma^{d+1}$, the eigenvalues of γ^5 and γ^{d+1} are interrelated. However the choice of the eigenvalue of Γ^{D+1} does not impose the eigenvalue on γ^5!

Majorana condition: $\psi = C\bar{\psi}^T$

Weyl-Majorana spinors can exist when $D = 4n + 2$.
Type of spinors of $SO(p, q)$ depends on signature $(p - q) mod 8$.
For $p + q = even$:

- 0: real rep
- 4: quaternionic rep
- 2 or 6: complex rep

Chapline & Slansky, 1982; Polchinski, 1998; D'Auria et al., 2001; Figueroa-O'Farrill, n.d.

Fermions (Continued)

In the case of $SO(2,16)$ the signature is 6, and imposing the Weyl and Majorana conditions is permitted.

Dirac spinors are defined as direct sum of Weyl spinors and the Weyl condition chooses one of them, say $\sigma_{18} = 256$.

Spinor rep branching rules are:

$$\begin{gathered} SO(18) \supset SU(4) \times SO(12) \\ 256 = (4, \bar{32}) + (\bar{4}, 32) \end{gathered}$$

Imposing Majorana condition the fermions are in the $(\bar{4}, 32)$. Then

$$\begin{gathered} SO(12) \supset SO(10) \times [U(1)] \\ 32 = (\bar{16})(1) + (16)(-1) \end{gathered}$$

On the other hand

$$\begin{gathered} SU(4) \to Sp_4 \to SU(2) \times SU(2) \\ 4 = 4 = (2,1) + (1,2). \end{gathered}$$

Fermions (Continued)

After all the breakings:

$$SU(2) \times SU(2) \times SO(10) \times [U(1)]$$

$$\{[(2,1) + (1,2)\}\{(16)(-1) + (\bar{16})(1)\}$$
$$= 16_L(-1) + \bar{16}_L(1) + 16_R(-1) + \bar{16}_R(1)$$

and since $\bar{16}_R(1) = 16_L(-1)$ and $\bar{16}_L(1) = 16_R(-1)$,

$$= 2 \times 16_L(-1) + 2 \times 16_R(-1).$$

Finally, keeping only the left-handed part we obtain:

$$2 \times 16_L(-1)$$

Imposing also the Majorana condition in lower dims we obtain

$$16_L(-1) \quad \text{of} \quad SO(10) \times [U(1)]$$

Fermions in Fuzzy Gravity and Unification with Internal Interactions

- Fermions should be chiral in the original theory to have a chance to survive in low energies
- they should appear in a matrix representation since FG is a matrix model

Fortunately the way out was suggested in unification schemes with extra fuzzy dimensions *Chatzistavrakidis, Steinacker, Z*

Instead of using fermions in fundamental, spinor or tensor reps of an $SU(N)$, we can use bi-fundamental reps of cross product $SU(N)$ groups.

Interesting example $N = 1$, $SU(N)^k$ models:

$$SU(N)_1 \times SU(N)_2 \times ... \times SU(N)^k$$

with matter content

$$(N, \bar{N}, 1, ..., 1) + (1, N, \bar{N}, ..., 1) + ... + (\bar{N}, 1, 1, ..., N)$$

Ma, Mondragon, Z, 2004

with successful phenomenology, $N = 1$, $SU(3)^3$.

Fermions in Fuzzy Gravity and Unification with Internal Interactions (Continued)

- In FG choosing to start with the $SO(6) \times SO(12)$ as the initial gauge theory with fermions in the $(4, \overline{32})$ we satisfy the criteria to obtain chiral fermions in tensorial representation.
- Weyl and Majorana conditions do not concern the global or local nature of the gauge part of the theory. Therefore all the discussion of unifying conformal conformal gravity with internal interactions can be repeated.
- The gauge $U(1)$ of FG due to the anticommutation relations, is identified with the one appearing in the $SO(12) \supset SO(10) \times U(1)$.

Thank you for your attention!