Tensor Gauge Field Theory
and
Extension of Chern-Simons Form

George Savvidy
Demokritos National Research Centre, Athens, Greece

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Publications

1. *Interaction of non-Abelian tensor gauge fields*
   Arm.J.Math. 1 (2008) 1-17 G.S.
2. *Extension of the Poincaré Group and Non-Abelian Tensor Gauge Fields*
   Int.J.Mod.Phys.A 25 (2010) 5765-5785 G.S
3. *Extensions of the Poincare group.*
   J.Math.Phys. 52 (2011) 072303 I.Antoniadis, L.Brink, G.S.

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4. *Topological mass generation in four-dimensional gauge theory.*
   Phys.Lett.B 694 (2011) 65-73 G.S.
5. *New gauge anomalies and topological invariants in various dimensions.*
   Eur.Phys.J.C 72 (2012) 2140 I.Antoniadis and G.S.
6. *Extension of Chern-Simons forms and new gauge anomalies.*
   Int.J.Mod.Phys.A 29 (2014) 1450027 I.Antoniadis and G.S.
7. *Extension of Chern-Simons forms.*
   J.Math.Phys. 55 (2014) 062304 S.Konitopoulos and G.S.
8. *Asymptotic freedom of non-Abelian tensor gauge fields.*
   Phys.Lett.B 732 (2014) 150-155 G.S.
9. *Generalisation of the Yang-Mills Theory.*
   Int.J.Mod.Phys.A 31 (2016) 1630003 G.S.
   International Conference on 60 Years of Yang-Mills Gauge Field Theories

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10. *Lars Brink Colleague, Friend and Collaborator* *G.S.*

Extension of the Poincaré Group

Representations and Killing Metric

Non-Abelian Tensor Gauge Fields

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The Lagrangian

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Topological Mass Generation

Topological invariants in various dimensions

Transgression and Secondary Forms

Extension of Chern-Simons Forms - CSAS

Extension of the Poincaré Algebra

$$[P^{\mu},\ P^{\nu}] = 0,$$
$$[M^{\mu\nu},\ P^{\lambda}] = i(\eta^{\lambda\nu}\ P^{\mu} - \eta^{\lambda\mu}\ P^{\nu}),$$
$$[M^{\mu\nu},\ M^{\lambda\rho}] = i(\eta^{\mu\rho}\ M^{\nu\lambda} - \eta^{\mu\lambda}\ M^{\nu\rho} + \eta^{\nu\lambda}\ M^{\mu\rho} - \eta^{\nu\rho}\ M^{\mu\lambda}),$$

$$[P^{\mu},\ L_a^{\lambda_1 \ldots \lambda_s}] = 0,$$
$$[M^{\mu\nu},\ L_a^{\lambda_1 \ldots \lambda_s}] = i(\eta^{\lambda_1\nu} L_a^{\mu\lambda_2\ldots\lambda_s} - \eta^{\lambda_1\mu} L_a^{\nu\lambda_2\ldots\lambda_s} + \ldots + \eta^{\lambda_s\nu} L_a^{\lambda_1\ldots\lambda_{s-1}\mu} - \eta^{\lambda_s\mu} L_a^{\lambda_1\ldots\lambda_{s-1}\nu}),$$

$$[L_a^{\lambda_1\ldots\lambda_i}, L_b^{\lambda_{i+1}\ldots\lambda_s}] = i f_{abc} L_c^{\lambda_1\ldots\lambda_s} \quad (\mu,\nu,\rho,\lambda = 0,1,2,3; \qquad s = 0,1,2,\ldots),$$

$$L_a^{\lambda_1\cdots\lambda_s} = e^{\lambda_1}\cdots e^{\lambda_s} \otimes L_a\,, \quad s = 0,1,2,\ldots .$$

These generators carry space–time and internal indices and transform under the operations of both groups. The algebra of these generators[6]

$$\left[L_a^{\lambda_1\cdots\lambda_i}, L_b^{\lambda_{i+1}\cdots\lambda_s}\right] = i f_{abc} L_c^{\lambda_1\cdots\lambda_s}\,, \quad s = 0,1,2,\ldots$$

Extension of the Poincaré Algebra

it is a "gauge invariant" extension of the Poincaré algebra in a sense that if one defines a "gauge" transformation of its generators as

$$L_a^{\lambda_1\cdots\lambda_s} \to L_a^{\lambda_1\cdots\lambda_s} + \sum_1 P^{\lambda_1} L_a^{\lambda_2\cdots\lambda_s} + \sum_2 P^{\lambda_1} P^{\lambda_2} L_a^{\lambda_3\cdots\lambda_s} + \cdots + P^{\lambda_1}\cdots P^{\lambda_s} L_a \,,$$

$$M^{\mu\nu} \to M^{\mu\nu}\,, \quad P^{\lambda} \to P^{\lambda}\,,$$

The algebra $L_G(\mathcal{P})$ has a simple representation of the following form

$$P^{\mu} = k^{\mu}\,,$$

$$M^{\mu\nu} = i\Big(k^{\mu}\frac{\partial}{\partial k_{\nu}} - k^{\nu}\frac{\partial}{\partial k_{\mu}}\Big) + i\Big(e^{\mu}\frac{\partial}{\partial e_{\nu}} - e^{\nu}\frac{\partial}{\partial e_{\mu}}\Big)\,,$$

$$L_a^{\lambda_1\cdots\lambda_s} = e^{\lambda_1}\cdots e^{\lambda_s} \otimes L_a\,,$$

$$k^2 = 0, \quad k^{\mu} e_{\mu} = 0, \quad e^2 = -1\,.$$

the matrix representations of this algebra are transversal

$$P_{\lambda_1} L_a^{\lambda_1\cdots\lambda_s} = 0\,.$$

Killing Metric

$$L_G : \qquad \langle L_a ; L_b \rangle = \delta_{ab},$$

$$L_{\mathcal{P}} : \qquad \langle P^\mu ; P^\nu \rangle = 0$$

$$\langle M_{\mu\nu} ; P_\lambda \rangle = 0$$

$$\langle M^{\mu\nu} ; M^{\lambda\rho} \rangle = \eta^{\mu\lambda}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\lambda}$$

$$L_G(\mathcal{P}) : \qquad \langle P^\mu ; L_a^{\perp\ \lambda_1 \ldots \lambda_s} \rangle = 0,$$

$$\langle M^{\mu\nu} ; L_a^{\perp\ \lambda_1 \ldots \lambda_s} \rangle = 0,$$

$$\langle L_a ; L_b^{\perp\ \lambda_1} \rangle = 0,$$

$$\langle L_a^{\perp\ \lambda_1} ; L_b^{\perp\ \lambda_2} \rangle = \delta_{ab}\ \bar{\eta}^{\lambda_1\lambda_2},$$

$$\langle L_a ; L_b^{\perp\ \lambda_1\lambda_2} \rangle = \delta_{ab}\ \bar{\eta}^{\lambda_1\lambda_2},$$

$$\langle L_a^{\perp\ \lambda_1} ; L_b^{\perp\ \lambda_2\lambda_3} \rangle = 0,$$

....

$$\langle L_a^{\perp\ \lambda_1 \ldots \lambda_n} ; L_b^{\perp\ \lambda_{n+1} \ldots \lambda_{2s+1}} \rangle = 0, \qquad s = 0, 1, 2, 3, \ldots$$

$$\langle L_a^{\perp\ \lambda_1 \ldots \lambda_n} ; L_b^{\perp\ \lambda_{n+1} \ldots \lambda_{2s}} \rangle = \delta_{ab}\ s!\ (\bar{\eta}^{\lambda_1\lambda_2}\bar{\eta}^{\lambda_3\lambda_4} \ldots \bar{\eta}^{\lambda_{2s-1}\lambda_{2s}} + \text{perm}),$$

where $\bar{\eta}^{\lambda_1\lambda_2}$ is the projector into the two-dimensional plane transversal to the momentum k^μ

$$\bar{\eta}^{\lambda_1\lambda_2} = \frac{k^{\lambda_1}\bar{k}^{\lambda_2} + \bar{k}^{\lambda_1}k^{\lambda_2}}{k\bar{k}} - \eta^{\lambda_1\lambda_2}, \qquad k_{\lambda_1}\bar{\eta}^{\lambda_1\lambda_2} = k_{\lambda_2}\bar{\eta}^{\lambda_1\lambda_2} = 0,$$

Non-Abelian Tensor Gauge Fields

The gauge fields are defined as rank-$(s+1)$ tensors

$$A^a_{\mu\lambda_1...\lambda_s}(x), \qquad s = 0, 1, 2, ...$$

and are totally symmetric with respect to the indices $\lambda_1...\lambda_s$. A priory the tensor fields have no symmetries with respect to the first index μ. The index a numerates the generators

L_a of the Lie algebra L_G of a compact Lie group G with totally antisymmetric structure constants f_{abc}.

$$\mathcal{A}_\mu(x,e) = \sum_{s=0}^{\infty} \frac{1}{s!} \; A^a_{\mu\lambda_1...\lambda_s}(x) \; L_a e^{\lambda_1}...e^{\lambda_s}.$$

The gauge field $A^a_{\mu\lambda_1...\lambda_s}$ carries indices $a, \lambda_1, ..., \lambda_s$ which are labelling the generators $L_a^{\lambda_1...\lambda_s} = L_a e^{\lambda_1}...e^{\lambda_s}$ of extended current algebra $L_{\mathcal{G}}$ associated with the Lie algebra

The gauge transformation of the field $\mathcal{A}_\mu(x,e)$ is defined as

$$\mathcal{A}'_\mu(x,e) = U(\xi)\mathcal{A}_\mu(x,e)U^{-1}(\xi) - \frac{\imath}{g}\partial_\mu U(\xi) \; U^{-1}(\xi),$$

Transformation of Tensor Gauge Fields

It is useful to have an explicit expression for the transformation law of the field components

$$
\begin{aligned}
\delta A^a_\mu &= (\delta^{ab}\partial_\mu + g f^{acb} A^c_\mu)\xi^b, \\
\delta A^a_{\mu\nu} &= (\delta^{ab}\partial_\mu + g f^{acb} A^c_\mu)\xi^b_\nu + g f^{acb} A^c_{\mu\nu}\xi^b, \\
\delta A^a_{\mu\nu\lambda} &= (\delta^{ab}\partial_\mu + g f^{acb} A^c_\mu)\xi^b_{\nu\lambda} + g f^{acb}(A^c_{\mu\nu}\xi^b_\lambda + A^c_{\mu\lambda}\xi^b_\nu + A^c_{\mu\nu\lambda}\xi^b),
\end{aligned}
$$

covariant derivatives $\nabla^{ab}_\mu = (\partial_\mu - ig\mathcal{A}_\mu(x,e))^{ab}$

$$[\nabla_\mu, \nabla_\nu]^{ab} = g f^{acb} \mathcal{G}^c_{\mu\nu} \ ,$$

field strengths tensors take the following form

$$
\begin{aligned}
G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}\ A^b_\mu\ A^c_\nu, \\
G^a_{\mu\nu,\lambda} &= \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} + g f^{abc}(\ A^b_\mu\ A^c_{\nu\lambda} + A^b_{\mu\lambda}\ A^c_\nu\ ), \\
G^a_{\mu\nu,\lambda\rho} &= \partial_\mu A^a_{\nu\lambda\rho} - \partial_\nu A^a_{\mu\lambda\rho} + g f^{abc}(\ A^b_\mu\ A^c_{\nu\lambda\rho} + A^b_{\mu\lambda}\ A^c_{\nu\rho} + A^b_{\mu\rho}\ A^c_{\nu\lambda} + A^b_{\mu\lambda\rho}\ A^c_\nu\ ) \\
\ldots\ldots &\quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Lagrangian of Tensor Gauge Fields

$$\mathcal{L}(x) = \langle \mathcal{L}(x,e) \rangle = -\frac{1}{4} \langle \mathcal{G}^a_{\mu\nu}(x,e) \mathcal{G}^{a\mu\nu}(x,e) \rangle,$$

$$\mathcal{L}(x,e) = \sum_{s=0}^{\infty} \frac{1}{s!} \mathcal{L}_{\lambda_1 ... \lambda_s}(x) \; e^{\lambda_1} ... e^{\lambda_s}.$$

$$\mathcal{L}(x) = \langle \mathcal{L}(x,e) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \mathcal{L}_{\lambda_1 ... \lambda_s}(x) \; \langle e^{\lambda_1} ... e^{\lambda_s} \rangle$$

using Killing metric

and the density for the lower-rank tensor fields is

$$\mathcal{L}_2 = -\frac{1}{4} G^a_{\mu\nu,\lambda} G^a_{\mu\nu,\lambda} - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu,\lambda\lambda}.$$

Lagrangian of Tensor Gauge Fields

$$\mathcal{L}^{'}(x) = \langle \mathcal{L}^{'}(x,e) \rangle = \frac{1}{4} \langle \mathcal{G}^{a}_{\mu\rho_1}(x,e) e^{\rho_1} \ \mathcal{G}^{a\mu} \ _{\rho_2}(x,e) e^{\rho_2} \rangle^{'}.$$

$$\mathcal{L}^{'}(x) = \langle \mathcal{L}^{'}(x,e) \rangle = \sum_{s=0}^{\infty} \ \frac{1}{s!} (\mathcal{L}^{'}_{\rho_1\rho_2})_{\lambda_1 ... \lambda_s}(x) \ \langle e^{\rho_1} e^{\rho_2} e^{\lambda_1} ... e^{\lambda_s} \rangle^{'}.$$

$$\mathcal{L}^{'}_2 = \frac{1}{4} G^{a}_{\mu\nu,\lambda} G^{a}_{\mu\lambda,\nu} + \frac{1}{4} G^{a}_{\mu\nu,\nu} G^{a}_{\mu\lambda,\lambda} + \frac{1}{2} G^{a}_{\mu\nu} G^{a}_{\mu\lambda,\nu\lambda}.$$

Lagrangian of Tensor Gauge Fields

$$L = \mathcal{L} + \mathcal{L}^{'} = -\frac{1}{4}\langle \mathcal{G}^{a}_{\mu\nu}(x,e)\mathcal{G}^{a\mu\nu}(x,e)\rangle + \frac{1}{4}\langle \mathcal{G}^{a}_{\mu\rho_1}(x,e)e^{\rho_1}\ \mathcal{G}^{a\mu}{}_{\rho_2}(x,e)e^{\rho_2}\rangle^{'}.$$

The Lagrangian for the lower-rank tensor gauge fields has the following form:

$$\begin{aligned}
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_2^{'} + ... = & - \frac{1}{4}G^{a}_{\mu\nu}G^{a}_{\mu\nu} \\
& - \frac{1}{4}G^{a}_{\mu\nu,\lambda}G^{a}_{\mu\nu,\lambda} - \frac{1}{4}G^{a}_{\mu\nu}G^{a}_{\mu\nu,\lambda\lambda} \\
& + \frac{1}{4}G^{a}_{\mu\nu,\lambda}G^{a}_{\mu\lambda,\nu} + \frac{1}{4}G^{a}_{\mu\nu,\nu}G^{a}_{\mu\lambda,\lambda} + \frac{1}{2}G^{a}_{\mu\nu}G^{a}_{\mu\lambda,\nu\lambda} + ...
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_3 + \mathcal{L}_3^{'} = & - \frac{1}{4}G^{a}_{\mu\nu,\lambda\rho}G^{a}_{\mu\nu,\lambda\rho} - \frac{1}{8}G^{a}_{\mu\nu,\lambda\lambda}G^{a}_{\mu\nu,\rho\rho} - \frac{1}{2}G^{a}_{\mu\nu,\lambda}G^{a}_{\mu\nu,\lambda\rho\rho} - \frac{1}{8}G^{a}_{\mu\nu}G^{a}_{\mu\nu,\lambda\lambda\rho\rho} + \\
& + \frac{1}{3}G^{a}_{\mu\nu,\lambda\rho}G^{a}_{\mu\lambda,\nu\rho} + \frac{1}{3}G^{a}_{\mu\nu,\nu\lambda}G^{a}_{\mu\rho,\rho\lambda} + \frac{1}{3}G^{a}_{\mu\nu,\nu\lambda}G^{a}_{\mu\lambda,\rho\rho} + \\
& + \frac{1}{3}G^{a}_{\mu\nu,\lambda}G^{a}_{\mu\lambda,\nu\rho\rho} + \frac{2}{3}G^{a}_{\mu\nu,\lambda}G^{a}_{\mu\rho,\nu\lambda\rho} + \frac{1}{3}G^{a}_{\mu\nu,\nu}G^{a}_{\mu\lambda,\lambda\rho\rho} + \frac{1}{3}G^{a}_{\mu\nu}G^{a}_{\mu\lambda,\nu\lambda\rho\rho}
\end{aligned} \tag{4.18}$$

Interaction of Tensor Gauge Fields

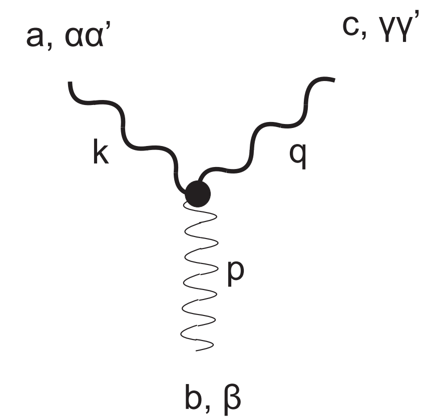

Figure 1: The interaction vertex for the vector gauge boson V and two tensor gauge bosons T - the VTT vertex - $\mathcal{V}^{abc}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q)$ in non-Abelian tensor gauge field theory [11]. Vector gauge bosons are conventionally drawn as thin wave lines, tensor gauge bosons are thick wave lines. The Lorentz indices $\alpha\acute{\alpha}$ and momentum k belong to the first tensor gauge boson, the $\gamma\acute{\gamma}$ and momentum q belong to the second tensor gauge boson, and Lorentz index β and momentum p belong to the vector gauge boson.

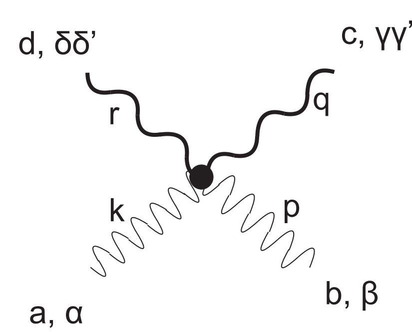

Figure 2: The quartic vertex with two vector gauge bosons and two tensor gauge bosons - the VVTT vertex - $\mathcal{V}^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r)$ in non-Abelian tensor gauge field theory [11]. Vector gauge bosons are conventionally drawn as thin wave lines, tensor gauge bosons are thick wave lines. The Lorentz indices $\gamma\acute{\gamma}$ and momentum q belong to the first tensor gauge boson, $\delta\acute{\delta}$ and momentum r belong to the second tensor gauge boson, the index α and momentum k belong to the first vector gauge boson and Lorentz index β and momentum p belong to the second vector gauge boson.

Callan-Simanzik Beta Function

$$b = \frac{(12s^2 - 1)C_2(G) - 4n_f T(R)}{12\pi}, \qquad s = 1, 2, ...$$

at s=1 we are rediscovering the asymptotic freedom

beta function has the same signature as the standard gluons, which means that tensorgluons "accelerate" the asymptotic freedom (6.3) of the strong interaction coupling constant $\alpha(t)$. The contribution is increasing quadratically with the spin of the tensorgluons, that is, at large transfer momentum the strong coupling constant tends to zero faster compared to the standard case:

$$\alpha(t) = \frac{\alpha}{1 + b\alpha\, t}\ , \tag{6.10}$$

Chern-Pontryagin density in 4-D Yang-Mills Theory

$$\mathcal{P}(A) = \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\,G_{\mu\nu}G_{\lambda\rho} = \partial_\mu C^\mu,$$

which is a derivative of the Chern–Simons topological vector current

$$C^\mu = \varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(A_\nu\partial_\lambda A_\rho - i\frac{2}{3}gA_\nu A_\lambda A_\rho\right).$$

Topological Mass Generation

Deser, Jackiw and Templeton and Schonfeld

who added to the YM Lagrangian a gauge invariant Chern-Simons density:

$$\mathcal{L}_{YMCS} = -\frac{1}{2} Tr G_{ij} G_{ij} + \frac{\mu}{2}\ \varepsilon_{ijk}\ Tr\ (A_i \partial_j A_k - ig\frac{2}{3} A_i A_j A_k),$$

where G_{ij} is a field strength tensor. The mass parameter μ carries dimension of $[mass]^1$. The corresponding free equation of motion for the vector potential $A_i = e_i e^{ikx}$ has the form

$$(-k^2 \eta_{ij} + k_i k_j) e_j + i\mu\ \varepsilon_{ijl}\ k_j e_l = 0$$

and shows that the gauge field excitation becomes massive.

we suggest a similar mechanism that generates masses of the YM boson

and tensor gauge bosons in (3+1)-dimensional space-time at the classical level.

New Topological Invariant in Tensor Gauge Theory

in non-Abelian tensor gauge theory there exists a gauge invariant, metric-independent density Γ in five-dimensional space-time[2]:

$$\Gamma = \varepsilon_{lmnpq} Tr G_{lm} G_{np,q} = \partial_l \Sigma_l,$$

which is the derivative of the vector current Σ_l (l,..=0,1,...,4).

$$\Sigma^l = \varepsilon^{lmnpq} \, \mathrm{Tr}(G_{mn} A_{pq}).$$

This invariant in five dimensions has many properties of the Chern–Pontryagin density

Γ is obviously diffeomorphism-invariant and does not involve a space–time metric.

It is gauge invariant because under the gauge transformation

$$\delta_\xi \Gamma = -ig\varepsilon_{\mu\nu\lambda\rho\sigma} \, \mathrm{Tr}\Big([G_{\mu\nu}\xi] G_{\lambda\rho,\sigma} + G_{\mu\nu}\big([G_{\lambda\rho,\sigma}\xi] + [G_{\lambda\rho}\xi_\sigma]\big)\Big) = 0.$$

New Topological Invariant in Tensor Gauge Theory

It became obvious that Γ is a total derivative of some vector current Σ_{μ}.

Indeed, simple algebraic computation gives $\Gamma = \varepsilon_{\mu\nu\lambda\rho\sigma} \operatorname{Tr} G_{\mu\nu} G_{\lambda\rho,\sigma} = \partial_{\mu} \Sigma_{\mu}$, where

$$\Sigma_{\mu} = 2\varepsilon_{\mu\nu\lambda\rho\sigma} \operatorname{Tr}(A_{\nu} \partial_{\lambda} A_{\varrho\sigma} - \partial_{\lambda} A_{\nu} A_{\rho\sigma} - 2ig A_{\nu} A_{\lambda} A_{\rho\sigma}).$$

After some rearrangement and taking into account the definition of the field strength tensors vector current:

$$\Sigma_{\mu} = \varepsilon_{\mu\nu\lambda\rho\sigma} \operatorname{Tr} G_{\nu\lambda} A_{\rho\sigma}.$$

Tensor Gauge Theory and Mass Generation

Let us consider the fifth component of the vector current Σ_μ:

$$\Sigma \equiv \Sigma_4 = \varepsilon_{4\nu\lambda\rho\sigma}\, \mathrm{Tr}\, G_{\nu\lambda} A_{\rho\sigma}\,.$$

the sum is restricted to the sum over indices of four-dimensional space–time.

This is the case when the gauge fields are independent on the fifth coordinate x_4.

integral over four-dimensional space–time[6]:

$$\int_{M_4} d^4x\, \Sigma = \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} d^4x\, \mathrm{Tr}\, G_{\nu\lambda} A_{\rho\sigma}\,.$$

As we claimed this functional is gauge invariant up to the total divergence term.

$$\delta_\xi \int_{M_4} d^4x\, \Sigma = \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} \mathrm{Tr}\Big(-ig[G_{\nu\lambda}\xi]A_{\rho\sigma} + G_{\nu\lambda}\big(\nabla_\rho \xi_\sigma - ig[A_{\rho\sigma}\xi]\big)\Big)\, d^4x$$

$$= \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} \partial_\rho\, \mathrm{Tr}(G_{\nu\lambda}\xi_\sigma)\, d^4x = \varepsilon_{\nu\lambda\rho\sigma} \int_{\partial M_4} \mathrm{Tr}(G_{\nu\lambda}\xi_\sigma)\, d\sigma_\rho = 0.$$

New gauge anomalies and topological invariants in various dimensions

Considering integral over four-dimensional space–time[2]

$$\Sigma(A) = \frac{1}{32\pi^2} \int_{M_4} d^4x\, \varepsilon^{\nu\lambda\rho\sigma}\, \mathrm{Tr}(G_{\nu\lambda} A_{\rho\sigma}).$$

This entity is an analog of the Chern–Simons integral[3]

$$W(A) = \frac{g^2}{8\pi^2} \int_{M_3} d^3x\, \varepsilon^{ijk}\, \mathrm{Tr}\Big(A_i \partial_j A_k - ig\frac{2}{3} A_i A_j A_k \Big),$$

but, *importantly, instead of being defined in three dimensions it is defined in four dimensions*. Thus, the non-Abelian tensor gauge fields allow to build a natural generalization of the Chern–Simons characteristic in four-dimensional space–time.

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The functional $\Sigma(A)$ is invariant under infinitesimal gauge transformations up to a total divergence term.

Tensor Gauge Theory and Mass Generation

In four dimensions the gauge fields have dimension of $[mass]^1$,

therefore if we intend to add this new density to the Lagrangian we

should introduce the mass parameter m:

$$m\Sigma = m\varepsilon_{\nu\lambda\rho\sigma} \,\mathrm{Tr}\, G_{\nu\lambda} A_{\rho\sigma} \,,$$

where parameter m has units $[mass]^1$.

we arrive at the following system of equations:

$$\partial^2 A_\nu - \partial_\nu \partial_\mu A_\mu + m\varepsilon_{\nu\mu\lambda\rho} \partial_\mu B_{\lambda\rho} = 0,$$

$$\partial^2 B_{\nu\lambda} - \partial_\nu \partial_\mu B_{\mu\lambda} + \partial_\lambda \partial_\mu B_{\mu\nu} + \frac{2m}{3} \varepsilon_{\nu\lambda\mu\rho} \partial_\mu A_\rho = 0.$$

antisymmetric part $B_{\nu\lambda}$ of the rank-2 gauge field $A_{\nu\lambda}$ interacts through the mass term, the symmetric part $A^{S}_{\nu\lambda}$ completely decouples

Tensor Gauge Theory and Mass Generation

$$(-k^2\eta_{\nu\mu} + k_\nu k_\mu)e_\mu + im\varepsilon_{\nu\mu\lambda\rho}k_\mu b_{\lambda\rho} = 0,$$

$$(-k^2\eta_{\nu\mu}\eta_{\lambda\rho} + k_\nu k_\mu \eta_{\lambda\rho} - \eta_{\nu\mu}k_\lambda k_\mu)b_{\mu\rho} + i\frac{2m}{3}\varepsilon_{\nu\lambda\mu\rho}k_\mu e_\rho = 0.$$

four pure gauge solutions $\quad e_\mu = k_\mu, \qquad b_{\nu\lambda} = 0;$

$$e_\mu = 0, \qquad b_{\nu\lambda} = k_\nu \xi_\lambda - k_\lambda \xi_\nu.$$

three solutions representing propagating modes:

$$e_\mu^{(1)} = (0,1,0,0), \qquad b_{\gamma\acute{\gamma}}^{(1)} = \frac{1}{i}\frac{M}{\sqrt{\vec{k}^2 + M^2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_\mu^{(2)} = (0,0,1,0), \qquad b_{\gamma\acute{\gamma}}^{(2)} = -\frac{1}{i}\frac{M}{\sqrt{\vec{k}^2 + M^2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$e_\mu^{(3)} = \left(0,0,0,\frac{M}{\sqrt{\vec{k}^2 + M^2}}\right), \qquad b_{\gamma\acute{\gamma}}^{(3)} = \frac{1}{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is a genuine superposition of vector and tensor fields.

Topological invariants in various dimensions

Considering integral over four-dimensional space–time[2]

$$\Sigma(A) = \frac{1}{32\pi^2} \int_{M_4} d^4x \, \varepsilon^{\nu\lambda\rho\sigma} \, \mathrm{Tr}(G_{\nu\lambda} A_{\rho\sigma}).$$

This entity is an analog of the Chern–Simons integral[3]

$$W(A) = \frac{g^2}{8\pi^2} \int_{M_3} d^3x \, \varepsilon^{ijk} \, \mathrm{Tr}\left(A_i \partial_j A_k - ig\frac{2}{3} A_i A_j A_k \right),$$

but, *importantly, instead of being defined in three dimensions it is defined in four dimensions*. Thus, the non-Abelian tensor gauge fields allow to build a natural generalization of the Chern–Simons characteristic in four-dimensional space–time.

The functional $\Sigma(A)$ is invariant under infinitesimal gauge transformations up to a total divergence term.

Large Gauge Transformations

$$A^U_\mu = U^- A_\mu U + \frac{i}{g}\, \underline{U^- \partial_\mu U},$$

$$A^U_{\mu\lambda} = U^- A_{\mu\lambda} U + U^- A_\mu U_\lambda - U^- U_\lambda U^- A_\mu U + \frac{i}{g}\, \underline{(U^- \partial_\mu U_\lambda - U^- U_\lambda U^- \partial_\mu U)},$$

where U_λ is the second term in the expansion of the unitary matrix $\mathcal{U}(\Xi(x,e))$ over the vector variable:

$$\mathcal{U}(x,e) = U(x) + U_\mu(x) e^\mu + ...,$$

$$\mathcal{U}^-(x,e) = U^-(x) - U^-(x) U_\mu(x) U^-(x) e^\mu + ...$$

$$U = 1 - igL_a\, \xi^a(x), \quad U_\mu = -igL_a\, \xi^a_\mu(x), ...$$

$$A^{flat}_\mu = \frac{i}{g}\, \underline{U^- \partial_\mu U}, \qquad A^{flat}_{\mu\lambda} = \frac{i}{g}\, \underline{(U^- \partial_\mu U_\lambda - U^- U_\lambda U^- \partial_\mu U)}.$$

Large Gauge Transformations

we have to find out how $\Sigma(A)$ transforms under large gauge transformations. The expression we found has the form :

$$\Sigma\left(A^U\right) - \Sigma(A) = \frac{i}{32\pi^2 g} \int_{M_4} d^4x\, \varepsilon^{\mu\nu\lambda\rho} \partial_\lambda \operatorname{Tr}\left(G_{\mu\nu} U_\rho U^-\right). \tag{9}$$

The expression (9) is analogous to the corresponding one of the Chern–Simons integral [

$$W\left(A^U\right) - W(A) = \frac{1}{8\pi^2} \int_{M_3} d^3x\, \varepsilon^{ijk} \partial_i \operatorname{Tr}\left(\partial_j U U^- A_k\right) + \frac{1}{24\pi^2} \int_{M_3} d^3x\, \varepsilon^{ijk} \operatorname{Tr}\left(U^- \partial_i U U^- \partial_j U U^- \partial_k U\right),$$

Topological invariants in Yang-Mills Theory

the divergence of the axial U(1) current $J^A_\mu = \psi\gamma_\mu\gamma_5\psi$,
in four dimensions it is given by the

$$\partial^\mu J^A_\mu = -\frac{1}{16\pi^2}\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}(G_{\mu\nu}G_{\lambda\rho})$$
$$= -\frac{1}{4\pi^2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu\,\mathrm{Tr}\left(A_\nu\partial_\lambda A_\rho - i\frac{2}{3}gA_\nu A_\lambda A_\rho\right).$$

Similarly, the non-Abelian anomaly appears in the covariant divergence of the non-Abelian left $J^{aL}_\mu = \bar{\psi}_L\gamma_\mu\gamma_5\sigma^a\psi_L$ or right $J^{aR}_\mu = \bar{\psi}_R\gamma_\mu\gamma_5\sigma^a\psi_R$ handed currents, such as

$$D^\mu J^{aL}_\mu = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu\ Tr[\sigma^a(A_\nu\partial_\lambda A_\rho - i\frac{1}{2}gA_\nu A_\lambda A_\rho)].$$

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the $U_A(1)$ anomaly is given by a $2n$-form, the higher-dimensional analog of Eq. (12):

$$d * J^A \propto \mathrm{Tr}\big(G^n\big) = d\omega_{2n-1},$$

where ω_{2n-1} is a generalization of the Chern–Simons form to $2n-1$ dimensions :

$$\omega_{2n-1}(A) = n\int_0^1 dt\,\mathrm{Tr}\big(AG_t^{n-1}\big).$$

$$A = -igA^a_\mu L_a\,dx^\mu,\ \text{with}\ G_t = tG + (t^2 - t)A^2.$$

Anomalies and Topological invariants in Yang-Mills Theory

gauge variation of the ω_{2n-1}:

$$\delta\omega_{2n-1} = d\omega^{1}_{2n-2}, \tag{21}$$

where the $(2n-2)$-form has the following integral representation

$$\omega^{1}_{2n-2}(\xi, A) = n(n-1)\int_0^1 dt\,(1-t)\,\mathrm{Str}\big(\xi\, d\big(A G_t^{n-2}\big)\big),$$

where $\xi = \xi^a L_a$ is a scalar gauge parameter and Str denotes a symmetrized trace. In $\mathcal{D} = 2n-2$ dimensions, the non-Abelian anomaly is given by this $(2n-2)$-form, the higher-dimensional analog :

$$D * J^{L,R}_{\xi} \propto \omega^{1}_{2n-2}(\xi, A).$$

New gauge anomalies in various dimensions

Our aim is to generalize the above construction by defining invariant densities in higher dimensions $\mathcal{D} = 2n + 3 = 5, 7, 9, 11, \ldots$:

$$\Gamma_{2n+3}(A) = \mathrm{Tr}\big(G^n G_3\big) = d\sigma_{2n+2},$$

where we are using a shorthand notation for the 3-form field-strength tensor $G_3 = dA_2 + [A, A_2]$ of the rank-2 gauge field $A_2 = -igA^a_{\mu\nu} L_a \, dx^\mu \wedge dx^\nu$ and $G_{3t} = tG_3 + (t^2 - t)[A, A_2]$. The $(2n+2)$-form σ_{2n+2} is

$$\sigma_{2n+2}(A, A_2) = \int_0^1 dt \, \mathrm{Tr}\big(A G_t^{n-1} G_{3t} + \cdots + G_t^{n-1} A G_{3t} + G_t^n A_2\big).$$

dimensionality of this density is $[mass]^{n(n+2)}$, and it can be used as an addition to the (2n+2)-dimensional Lagrangian density

$$\frac{1}{F^{n^2-2}} \int_{M_{2n+2}} \sigma_{2n+2}(A, A_2),$$

Topological invariants in various dimensions

We also found a second series of exact $6n$-forms constructed only in terms of the 3-form gauge field-strength G_3:

$$\Delta_{6n} = \mathrm{Tr}(G_3)^{2n} = d\pi_{6n-1},$$

where for the $(6n-1)$-form one gets the following expression:

$$\pi_{6n-1}(A, A_2) = 2n \int_0^1 dt\, \mathrm{Tr}\big(A_2 G_{3t}^{2n-1}\big).$$

These forms are defined in $\mathcal{D} = 6n - 1 = 5, 11, 17, \ldots$ dimensions.

Our next aim is to construct possible gauge anomalies σ^1_{2n+1} and π^1_{6n-2} which follow from the generalized densities σ_{2n+2} and π_{6n-1}. These potential anomalies are defined through the relation analogous to (21):

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$$\delta\sigma_{2n+2} = d\sigma^1_{2n+1}, \qquad \delta\pi_{6n-1} = d\pi^1_{6n-2}.$$

New gauge anomalies and topological invariants in various dimensions

$$\sigma_3^1(\xi_1, A) = \mathrm{Tr}(\xi_1 G),$$

$$\sigma_5^1(\xi_1, A) = \mathrm{Tr}\left(\xi_1\, d\left(A\, dA + \frac{1}{2} A^3\right)\right),$$

where $\xi_1 = \xi_\mu^a L_a\, dx^\mu$ is a 1-form gauge parameter

and when the gauge transformation is performed by a scalar gauge parameter ξ, then

$$\sigma_5^1(\xi, A, A_2) = \mathrm{Tr}\Big(\xi\, d\Big(A\, dA_2 + A_2\, dA + \frac{1}{2} A^2 A_2 - \frac{1}{2} A A_2 A + \frac{1}{2} A_2 A^2\Big)\Big).$$

its descendant $(\delta\sigma_5^1 = d\sigma_4^2)$

$$\sigma_4^2(\xi, \eta, A) = \mathrm{Tr}\big((d\xi\, \eta + \eta\, d\xi - \xi\, d\eta - d\eta\, \xi)\, dA_2\big)$$

may represent a potential Schwinger term in the corresponding gauge algebra .

New gauge anomalies - transgression

In conclusion let us compare the Pontryagin–Chern–Simons densities $\mathcal{P}_{2n}$, ω_{2n-1} and ω^1_{2n-2} in YM gauge theory with the corresponding densities Γ_{2n+3}, σ_{2n+2}, σ^1_{2n+1} and Δ_{6n}, π_{6n-1}, π^1_{6n-2} in the extended YM theory. The new characteristic classes are local forms defined on the space–time manifold and constructed from the curvature 2-form G and 3-form G_3:

$$\Gamma_{2n+3} = \mathrm{Tr}\big(G^n G_3\big) = d\sigma_{2n+2},$$

$$\Delta_{6n} = \mathrm{Tr}(G_3)^{2n} = d\pi_{6n-1}.$$

the existence of these potential anomalies is based on the fact that they fulfill Wess–Zumino consistency conditions. At the same time, these invariant densities constructed on the space–time manifold have their own independent value since they suggest the existence of new invariants characterizing topological properties of a manifold.

New Topological Field Theories

$$Z(M, M_2^i, C^j, R) = \int \mathcal{D}A\mathcal{D}A_2 e^{ik\int_M \sigma_{2n+2}(A,A_2)} \prod_{i,j} Tr_{R_i} e^{i\oint_{M_2^i} A_2} Tr_{R_j} e^{i\oint_{C^j} A},$$

where σ_{2n+2} is defined in (1.6) and k is a parameter, or on three-dimensional manifolds

$$Z(M, M_3^i, C^j, R) = \int \mathcal{D}A\mathcal{D}A_3 e^{ik\int_M \psi_{2n+3}(A,A_3)} \prod_{i,j} Tr_{R_i} e^{i\oint_{M_3^i} A_3} Tr_{R_j} e^{i\oint_{C^j} A}$$

as well as on higher dimensional ones, ψ_{2n+3} is defined in (1.8). In particular, for the partition function $Z(M)$ in four dimensions we get

$$Z(M) = \int \mathcal{D}A\mathcal{D}A_2 e^{ik\int_{M_4} \sigma_4} = \int \mathcal{D}A\mathcal{D}A_2 e^{ik\int_{M_4} Tr(GA_2)}$$

and in the large k limit the contribution to the path integral is dominated from the points of stationary phase which are, in the given case, the flat connections

$$G = dA + A^2 = 0, \qquad G_3 = dA_2 + [A, A_2] = 0.$$

$$\mathcal{P}_{2n} \;\Rightarrow\; \omega_{2n-1} \;\Rightarrow\; \omega^1_{2n-2}.$$

Therefore we shall perform the following transgressions:

$$\Phi_{2n+4} \;\Rightarrow\; \psi_{2n+3} \;\Rightarrow\; \psi^1_{2n+2},$$

$$\Xi_{2n+6} \;\Rightarrow\; \phi_{2n+5} \;\Rightarrow\; \phi^1_{2n+4},$$

$$\Upsilon_{2n+8} \;\Rightarrow\; \rho_{2n+7} \;\Rightarrow\; \rho^1_{2n+6}.$$

Thank You !

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