Algebraic Bethe Ansatz for Gaudin model in SO(3) case

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Talk outline

- A few words on motivation
- A few words on Gaudin model in general
- SO(3) Gaudin model with boundary
  - Preliminaries (r-matrix, Lax operator)
  - Nontrivial “boundary” conditions
  - Gaudin Hamiltonians
  - General boundary -> special vacuum
  - Solving ABA
- Knizhnik-Zamolodchikov equations
- Norm and scalar product formulas

Motivation?

- So few problems that we can exactly solve
  - Any such is precious
  - The more general, the better
  - Expected to provide insights to realistic problems
  - Gaudin model – integrable, with long range interactions

Gaudin model

- Introduced as a “quasi-classical” expansion of the spin-chain models, for example:

$$R(\lambda) = \lambda 1 + \eta \mathscr{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}$$

quasi-classical parameter

$$\frac{1}{\lambda} R(\lambda) = 1 - \eta r(\lambda)$$

classical r-matrix

Yang-Baxter

$$R_{12}(\lambda-\mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda-\mu)$$

“Classical” Yang-Baxter

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)] = 0$$

Gaudin model - independent approach

- Classical r-matrix:

$$[r_{32}(\lambda_3,\lambda_2), r_{13}(\lambda_1,\lambda_3)] + [r_{12}(\lambda_1,\lambda_2), r_{13}(\lambda_1,\lambda_3) + r_{23}(\lambda_2,\lambda_3)] = 0$$

- Lax operator satisfies:

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = [r_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)] - [r_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)]$$

- Then, if we define generating function:

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda)$$

- It will commute for different values of “spectral parameters”:

$$[\tau(\lambda), \tau(\mu)] = 0$$

So what?

- For a given r-matrix, we can find “local” realization of Lax operator that acts in a multiparticle Hilbert space $\mathcal{H} = \bigotimes_{m=1}^{N} V_m$
- Generating function, or its derivatives (residues) can take form of Hamiltonians with nontrivial interactions
- Solving eigenproblem of the generating function $\tau(\lambda)$ provides all relevant information about system (energies, Hamiltonian eigenstates, integrals of motion)
- There is an efficient way to solve $\tau(\lambda)$ eigenproblem, know as Algebraic Bethe Ansatz (ABA)
- Gaudin model exhibits long-range interactions

Rational so(3) Gaudin model

- We start from the classical r-matrix:

$$r_{12}(\lambda) = -\frac{\vec{\mathbb{S}}_1 \cdot \vec{\mathbb{S}}_2}{\lambda} \quad \mathbb{S}^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathbb{S}^y = \frac{\imath}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathbb{S}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- It corresponds to Lax operator:

arbitrary "inhomogeneous" parameters

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\vec{\mathbb{S}}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} = \sum_{m=1}^{N} \frac{1}{\lambda - \alpha_m}\left(\mathbb{S}_0^3 \otimes S_m^3 + \frac{1}{2}\left(\mathbb{S}_0^+ \otimes S_m^- + \mathbb{S}_0^- \otimes S_m^+\right)\right)$$

- where spin operators $S_m^\alpha, \alpha = +, -, 3$ live in $\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}$

and satisfy: $[S_m^3, S_n^\pm] = \pm S_m^\pm \delta_{mn}, \quad [S_m^+, S_n^-] = 2S_m^3 \delta_{mn}$

- Tightly related to rational SO(3) Heisenberg spin-chain model. Lax operator there has form:

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{2\eta}{\lambda}\vec{\mathbb{S}}_0 \cdot \vec{S}_m + \frac{2\eta^2}{\lambda(\lambda+\eta)}\left(\left(\vec{\mathbb{S}}_0 \cdot \vec{S}_m\right)^2 - \mathbb{1}\right)$$

Rational so(3) Gaudin model

- It holds: $[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)] = 0,$ $\quad r_{21}(-\lambda) = -r_{12}(\lambda)$

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda-\mu), L_1(\lambda) + L_2(\mu)]$$

- So that, if we define: $\tau(\lambda) = \frac{1}{2} \mathrm{tr}\, L^2(\lambda)$
- it will commute for different values of the spectral parameter:

$$[\tau(\lambda), \tau(\mu)] = 0$$

- It will also commute with its residue, that has the form of a Hamiltonian:

$$\operatorname*{Res}_{\lambda=\alpha_m} \tau(\lambda) = 4 H_m \qquad \mathrm{H}_m = \sum_{n \neq m}^{N} \frac{\vec{S}_n \vec{S}_m}{\alpha_m - \alpha_n}$$

long-range interaction

Introducing nontrivial “boundary”

- Name comes from the relation to the boundary terms of a spin-chain
- Model can be generalized by a K matrix satisfying “reflection equation”:

$$r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu)+K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)=$$
$$=K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)+K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu).$$

- SO(3) solutions can be obtained using “fusion procedure”:

$$K(\lambda)=\begin{pmatrix} (\xi-\nu\lambda)^2 & -\sqrt{2}\,\psi\,\lambda(\xi-\nu\lambda) & \psi^2\lambda^2 \\ -\sqrt{2}\,\varphi\lambda(\xi-\nu\lambda) & \xi^2+(\psi\varphi-\nu^2)\,\lambda^2 & -\sqrt{2}\,\psi\lambda(\xi+\nu\lambda) \\ \varphi^2\lambda^2 & -\sqrt{2}\,\varphi\lambda(\xi+\nu\lambda) & (\xi+\nu\lambda)^2 \end{pmatrix}$$

- Commonly, we fix some of the parameters to zero, to have ground state for algebraic Bethe ansatz
- This time we do better!

Nontrivial “boundary”

- Now, we can define more general:

$$r^K_{00'}(\lambda,\mu) = r_{00'}(\lambda-\mu) - K_{0'}(\mu) r_{00'}(\lambda+\mu) K^{-1}_{0'}(\mu)$$

$$\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda) L_0(-\lambda) K^{-1}_0(\lambda)$$

- that again satisfy:

$$[r^K_{32}(\lambda_3,\lambda_2), r^K_{13}(\lambda_1,\lambda_3)] + [r^K_{12}(\lambda_1,\lambda_2), r^K_{13}(\lambda_1,\lambda_3) + r^K_{23}(\lambda_2,\lambda_3)] = 0$$

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right]$$

- so we define more general generating fuction

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}^2_0(\lambda) \quad \Rightarrow \quad [\tau(\lambda), \tau(\mu)] = 0$$

Gaudin Hamiltonians

long-range interaction

- We take residues of generating function:

$$\operatorname*{Res}_{\lambda=\alpha_m} \tau(\lambda) \;=\; 4\, H_m$$

- Result has a form of "tunable" many-particle interacting Hamiltonian-like operators:

By solving eigenproblem of generating function we also find energies and H. eigenstates!

$$\mathrm{H}_m = (\pm) \operatorname*{Res}_{\lambda=\pm\alpha_m} \tau(\lambda) = \frac{\cdots}{\xi^2 - \cdots} \cdots \frac{\alpha_m}{2} \left(\psi^2 \left(S_m^+\right)^2 + \varphi^2 \left(S_m^-\right)^2\right.$$

$$\left. +2\psi\nu \left(S_m^+ S_m^3 + S_m^3 S_m^+\right) + 2\varphi\nu \left(S_m^- S_m^3 + S_m^3 S_m^-\right)\right) + \frac{\xi^2 + \nu^2\alpha_m^2}{2\alpha_m} \left(S_m^+ S_m^- + S_m^- S_m^+\right)\Big)$$

$$+ \frac{\alpha_m}{\xi^2 - (\psi\varphi + \nu^2)\,\alpha_m^2} \sum_{n\neq m}^{N} \left(\frac{4\left(\xi^2 - \psi\varphi\,\alpha_m\alpha_n - \nu^2\alpha_m^2\right)}{\alpha_m^2 - \alpha_n^2} S_m^3 S_n^3 - \frac{\alpha_m}{\alpha_m + \alpha_n} \left(\psi^2\, S_m^+ S_n^+ + \varphi^2\, S_m^- S_n^-\right.\right.$$

$$\left. +2\psi\nu \left(S_m^+ S_n^3 + S_m^3 S_n^+\right) + 2\varphi\nu \left(S_m^- S_n^3 + S_m^3 S_n^-\right)\right) + \frac{2\left(\xi^2 - (\psi\varphi + \nu^2)\,\alpha_m\alpha_n\right) - \psi\varphi\,\alpha_m\left(\alpha_m - \alpha_n\right)}{\alpha_m^2 - \alpha_n^2} \times$$

$$\times \left(S_m^- S_n^+ + S_m^+ S_n^-\right)\Big)$$

"boundary" parameters

$$+ \frac{\xi \cdot \alpha_m}{\xi^2 - (\psi\varphi + \nu^2)\,\alpha_m^2} \sum_{n\neq m}^{N} \frac{1}{\alpha_m + \alpha_n} \left(2\psi \left(S_m^+ S_n^3 - S_m^3 S_n^+\right) + 2\nu \left(S_m^- S_n^+ - S_m^+ S_n^-\right) + 2\varphi \left(S_m^3 S_n^- - S_m^- S_n^3\right)\right)$$

Algebraic Bethe Ansatz

- Writing: $\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda) L_0(-\lambda) K_0^{-1}(\lambda) = \begin{pmatrix} H(\lambda) & \frac{1}{\sqrt{2}} F(\lambda) & 0 \\ \frac{1}{\sqrt{2}} E(\lambda) & 0 & \frac{1}{\sqrt{2}} F(\lambda) \\ 0 & \frac{1}{\sqrt{2}} E(\lambda) & -H(\lambda) \end{pmatrix}$
- from: $[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = [r_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)] - [r_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)]$
- we obtain:

$$[E(\lambda), E(\mu)] = \frac{-2\varphi^2}{\lambda+\mu}\left(\frac{\mu^2}{\xi^2-(\psi\varphi+\nu^2)\mu^2} H(\lambda) - \frac{\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} H(\mu)\right),$$
$$+ \frac{2\varphi}{\lambda+\mu}\left(\frac{(\xi+\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} E(\lambda) - \frac{(\xi+\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda} E(\mu)\right),$$

$$[F(\lambda), F(\mu)] = \frac{2\psi^2}{\lambda+\mu}\left(\frac{\mu^2}{\xi^2-(\psi\varphi+\nu^2)\mu^2} H(\lambda) - \frac{\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} H(\mu)\right)$$
$$+ \frac{2\psi}{\lambda+\mu}\left(\frac{(\xi-\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} F(\lambda) - \frac{(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda} F(\mu)\right),$$

$$[H(\lambda), H(\mu)] = \frac{-\psi}{\lambda+\mu}\left(\frac{(\xi+\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2} E(\lambda) - \frac{(\xi+\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} E(\mu)\right)$$
$$+ \frac{-\varphi}{\lambda+\mu}\left(\frac{(\xi-\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} F(\lambda) - \frac{(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} F(\mu)\right),$$

$$[H(\lambda), E(\mu)] = \frac{\varphi}{\lambda+\mu}\left(\frac{\varphi\,\mu^2}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2} F(\lambda) - \frac{2\,(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} H(\mu)\right)$$
$$- \frac{1}{(\lambda-\mu)(\lambda+\mu)}\left(\frac{(2(\xi-\nu\lambda)\,(\xi+\nu\mu) - \psi\varphi\,(\lambda+\mu)\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} E(\lambda)\right.$$
$$\left. - \frac{2\left(\xi^2-\left(\psi\varphi\,\mu+\nu^2\lambda\right)\lambda\right)\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} E(\mu)\right),$$

$$[H(\lambda), F(\mu)] = \frac{-\psi}{\lambda+\mu}\left(\frac{\psi\,\mu^2}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2} E(\lambda) + \frac{2\,(\xi+\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} H(\mu)\right)$$
$$+ \frac{1}{(\lambda-\mu)(\lambda+\mu)}\left(\frac{(2(\xi-\nu\mu)\,(\xi+\nu\lambda) - \psi\varphi\,(\lambda+\mu)\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} F(\lambda)\right.$$
$$\left. - \frac{2\left(\xi^2-\left(\psi\varphi\,\mu+\nu^2\lambda\right)\lambda\right)\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} F(\mu)\right),$$

$$[F(\lambda), E(\mu)] = \frac{-2}{\lambda+\mu}\left(\frac{\varphi\,(\xi+\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2} F(\lambda) - \frac{\psi\,(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} E(\mu)\right)$$
$$+ \frac{2}{(\lambda-\mu)(\lambda+\mu)}\left(\frac{(2(\xi-\nu\lambda)\,(\xi+\nu\mu) - \psi\varphi\,(\lambda+\mu)\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} H(\lambda)\right.$$
$$\left. - \frac{(2(\xi-\nu\lambda)\,(\xi+\nu\mu) - \psi\varphi\,(\lambda+\mu)\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} H(\mu)\right).$$

Terrible!?!

Wise change of basis

- Define new operators:

$$\mathcal{E}(\lambda) = \frac{1}{2\psi\sqrt{\psi\varphi+\nu^2}} \left(\psi^2 E(\lambda) - \left(\psi\varphi + 2\nu\left(\nu - \sqrt{\psi\varphi+\nu^2}\right)\right) F(\lambda) + 2\psi\left(\nu - \sqrt{\psi\varphi+\nu^2}\right) H(\lambda)\right)$$

$$\mathcal{F}(\lambda) = \frac{1}{2\psi\sqrt{\psi\varphi+\nu^2}} \left(-\psi^2 E(\lambda) + \left(\psi\varphi + 2\nu\left(\nu + \sqrt{\psi\varphi+\nu^2}\right)\right) F(\lambda) - 2\psi\left(\nu + \sqrt{\psi\varphi+\nu^2}\right) H(\lambda)\right)$$

$$\mathcal{H}(\lambda) = \frac{1}{2\sqrt{\psi\varphi+\nu^2}} \left(\psi E(\lambda) + \varphi F(\lambda) + 2\nu H(\lambda)\right)$$

- Now: $[\mathcal{E}(\lambda), \mathcal{E}(\mu)] = [\mathcal{F}(\lambda), \mathcal{F}(\mu)] = [\mathcal{H}(\lambda), \mathcal{H}(\mu)] = 0\,,$

$$[\mathcal{H}(\lambda), \mathcal{E}(\mu)] = \frac{-2}{\lambda^2-\mu^2}\left(\mu\,\frac{\xi - \lambda\sqrt{\psi\varphi+\nu^2}}{\xi - \mu\sqrt{\psi\varphi+\nu^2}}\,\mathcal{E}(\lambda) - \lambda\,\mathcal{E}(\mu)\right),$$

$$[\mathcal{H}(\lambda), \mathcal{F}(\mu)] = \frac{2}{\lambda^2-\mu^2}\left(\mu\,\frac{\xi + \lambda\sqrt{\psi\varphi+\nu^2}}{\xi + \mu\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda) - \lambda\,\mathcal{F}(\mu)\right),$$

$$[\mathcal{F}(\lambda), \mathcal{E}(\mu)] = \frac{4}{\lambda^2-\mu^2}\left(\mu\,\frac{\xi - \lambda\sqrt{\psi\varphi+\nu^2}}{\xi - \mu\sqrt{\psi\varphi+\nu^2}}\,\mathcal{H}(\lambda) - \lambda\,\frac{\xi + \mu\sqrt{\psi\varphi+\nu^2}}{\xi + \lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{H}(\mu)\right).$$

Local realization

$$\mathcal{E}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi + \nu^2}} \sum_{m=1}^{N} \frac{\xi - \alpha_m\sqrt{\psi\varphi + \nu^2}}{\xi - \lambda\sqrt{\psi\varphi + \nu^2}}$$

$$\times \frac{2(\nu - \sqrt{\psi\varphi + \nu^2})S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2\nu\,(\nu - \sqrt{\psi\varphi + \nu^2})}{\psi} S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)} \, ,$$

$$\mathcal{F}(\lambda) = \frac{-\lambda}{\sqrt{\psi\varphi + \nu^2}} \sum_{m=1}^{N} \frac{\xi + \alpha_m\sqrt{\psi\varphi + \nu^2}}{\xi + \lambda\sqrt{\psi\varphi + \nu^2}}$$

$$\times \frac{2(\nu + \sqrt{\psi\varphi + \nu^2})S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2\nu\,(\nu + \sqrt{\psi\varphi + \nu^2})}{\psi} S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)} \, ,$$

$$\mathcal{H}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi + \nu^2}} \sum_{m=1}^{N} \frac{2\nu S_m^3 + \psi S_m^+ + \varphi S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)} \, .$$

“Vacuum” eigenstate

- Generating function is:

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda) = \mathcal{H}^2(\lambda) + \frac{1}{2}\left(\mathcal{E}(\lambda)\mathcal{F}(\lambda) + \mathcal{F}(\lambda)\mathcal{E}(\lambda)\right)$$

- “All-spins-up” state Ω_+ is an eigen&vacuum state:

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H} \qquad S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0$$

- since $\mathcal{E}(\lambda)\,\Omega_+ = 0 \qquad \mathcal{H}(\lambda)\,\Omega_+ = \rho(\lambda)\,\Omega_+ \qquad \tau(\lambda)\,\Omega_+ = \chi_0(\lambda)\,\Omega_+$
- …

But it is not!!!

Unless we reduce generality and set *y*=0 ???...

New (general) vacuum state

- Define:

$$\omega_m = \begin{pmatrix} \psi^2 \\ -\sqrt{2}\,\psi\left(\nu - \sqrt{\psi\varphi + \nu^2}\right) \\ \left(\nu - \sqrt{\psi\varphi + \nu^2}\right)^2 \end{pmatrix} \in \mathbb{C}^3 = V_m \qquad \Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}$$

- so that:

$$\left(2\left(\nu - \sqrt{\psi\varphi + \nu^2}\right) S_m^3 + \psi\; S_m^+ - \frac{\psi\varphi + 2\nu\left(\nu - \sqrt{\psi\varphi + \nu^2}\right)}{\psi}\; S_m^-\right)\omega_m = 0\,,$$

$$\left(2\nu S_m^3 + \psi S_m^+ + \varphi S_m^-\right)\omega_m = 2\sqrt{\psi\varphi + \nu^2}\;\omega_m\,.$$

- i.e. so that now really:

$$\mathcal{E}(\lambda)\;\Omega_+ = 0 \quad \text{and} \quad \mathcal{H}(\lambda)\;\Omega_+ = \rho(\lambda)\;\Omega_+ \quad \text{with} \quad \rho(\lambda) = \sum_{m=1}^{N} \frac{2\lambda}{\lambda^2 - \alpha_m^2}$$

$$\tau(\lambda)\,\Omega_+ = \chi_0(\lambda)\,\Omega_+ \quad \text{with} \quad \chi_0(\lambda) = \rho^2(\lambda) + \frac{\xi^2 + (\psi\varphi + \nu^2)\lambda^2}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2}\,\frac{\rho(\lambda)}{\lambda} - \rho'(\lambda)$$

To get rid of “unwanted terms” we impose Bethe equations:

$$\rho(\mu_j) + \frac{(\psi\varphi + \nu^2)\mu_j}{\xi^2 - (\psi\varphi + \nu^2)\mu_j^2} - \sum_{k \neq j}^{M} \frac{2\mu_j}{\mu_j^2 - \mu_k^2} = 0$$

- We sho … s!
- General solution can be written as:

$$\Phi_M(\mu_1, \mu_2, \ldots, \mu_M) = \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\Omega_+ \,,$$

- since:

$$\tau(\lambda)\Phi_M(\mu_1, \mu_2, \ldots, \mu_M) = \chi_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M)\ \Phi_M(\mu_1, \mu_2, \ldots, \mu_M)$$

$$+ \sum_{j=1}^{M} \frac{4\lambda}{\lambda^2 - \mu_j^2} \frac{\xi - \mu_j\sqrt{\psi\varphi + \nu^2}}{\xi - \lambda\sqrt{\psi\varphi + \nu^2}} \left(\rho(\mu_j) + \frac{(\psi\varphi + \nu^2)\mu_j}{\xi^2 - (\psi\varphi + \nu^2)\mu_j^2} - \sum_{k \neq j}^{M} \frac{2\mu_j}{\mu_j^2 - \mu_k^2} \right) \times$$

$$\times\ \Phi_M(\lambda, \mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M) \,,$$

- eigenvalues:

$$\chi_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M) = \chi_0(\lambda) - \sum_{j=1}^{M} \frac{4\lambda}{\lambda^2 - \mu_j^2} \left(\rho(\lambda) + \frac{(\psi\varphi + \nu^2)\lambda}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2} - \sum_{k \neq j}^{M} \frac{\lambda}{\lambda^2 - \mu_k^2} \right) .$$

Action of Gaudin Hamiltonians

- Full off-shell action:

$$\mathrm{H}_m\,\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) = \mathcal{E}_{m,M}\,\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) + \sum_{j=1}^{M}\frac{4\alpha_m}{\alpha_m^2-\mu_j^2}\,\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\alpha_m\sqrt{\psi\varphi+\nu^2}}\times$$

$$\times\left(\rho(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j}{\mu_j^2-\mu_k^2}\right)\left(\frac{-2(\nu+\sqrt{\psi\varphi+\nu^2})S_m^3-\psi S_m^+ + \frac{\psi\varphi+2\nu\,(\nu+\sqrt{\psi\varphi+\nu^2})}{\psi}S_m^-}{2\sqrt{\psi\varphi+\nu^2}}\right)\times$$

$$\times\,\Phi_{M-1}(\mu_1,\ldots,\widehat{\mu}_j,\ldots,\mu_M)\,,$$

- energies:

$$\mathcal{E}_{m,M} = \operatorname*{Res}_{\lambda=\alpha_m}\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)$$

$$= \frac{2\xi^2}{\left(\xi^2-(\psi\varphi+\nu^2)\,\alpha_m^2\right)\alpha_m} + \sum_{n\neq m}^{N}\frac{4\alpha_m}{\alpha_m^2-\alpha_n^2} - \sum_{j=1}^{M}\frac{4\alpha_m}{\alpha_m^2-\mu_j^2}$$

Knizhnik-Zamolodchikov equations

- We want to find $\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ such that:

$$\kappa\, \partial_{\alpha_m} \Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \widetilde{H}_m\, \Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)$$

- Must take: $\xi = 0$
- Seek solutions in form:

$$\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \oint \oint \cdots \oint \Upsilon(\vec{\mu}\,; \vec{\alpha}) \cdot \widetilde{\Phi}_M(\vec{\mu}\,; \vec{\alpha})\, \mathrm{d}\mu_1\, \mathrm{d}\mu_2 \cdots \mathrm{d}\mu_M$$

- where:

$$\kappa\, \partial_{\alpha_m} \Upsilon = \widetilde{\mathcal{E}}_{m,M}\, \Upsilon\,,$$

$$\kappa\, \partial_{\mu_j} \Upsilon = \beta_M(\mu_j)\, \Upsilon\,,$$

$$\beta_M(\mu_j) = -2\left(\rho(\mu_j) - \frac{1}{\mu_j} - \sum_{k \neq j}^{M} \frac{2\mu_j}{\mu_j^2 - \mu_k^2}\right)$$

We find:

$$\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \oint \oint \cdots \oint \Upsilon\left(\overrightarrow{\mu}\,;\overrightarrow{\alpha}\right) \cdot \widetilde{\Phi}_M\left(\overrightarrow{\mu}\,;\overrightarrow{\alpha}\right) \mathrm{d}\mu_1\, \mathrm{d}\mu_2 \cdots \mathrm{d}\mu_M$$

Where:

$$\Upsilon\left(\overrightarrow{\mu}\,;\overrightarrow{\alpha}\right) = \exp\left(\frac{S(\overrightarrow{\mu}\,;\overrightarrow{\alpha})}{\kappa}\right)$$

$$S\left(\overrightarrow{\mu}\,;\overrightarrow{\alpha}\right) = \sum_{m=1}^{N}\left(\sum_{n\neq m}^{N}\ln\left(\alpha_n^2-\alpha_m^2\right) - \sum_{j=1}^{M} 2\ln\left(\mu_j^2-\alpha_m^2\right)\right) + \sum_{j=1}^{M}\left(\ln\left(\mu_j^2\right) + \sum_{k\neq j}^{M}\ln\left(\mu_j^2-\mu_k^2\right)\right).$$

Also, neat formulas for…

…norms (on-shell):

$$\left\|\widetilde{\Phi}_M(\mu_1,\mu_2,\ldots,\mu_M)\right\|^2 = \det\begin{pmatrix} \frac{\partial^2 S}{\partial\mu_1^2} & \frac{\partial^2 S}{\partial\mu_1\partial\mu_2} & \cdots & \frac{\partial^2 S}{\partial\mu_1\partial\mu_M} \\ \vdots & \ddots & & \vdots \\ \frac{\partial^2 S}{\partial\mu_M\partial\mu_1} & \frac{\partial^2 S}{\partial\mu_M\partial\mu_2} & \cdots & \frac{\partial^2 S}{\partial\mu_M^2} \end{pmatrix}\Bigg|_{\substack{\beta_M(\mu_1)=0\\ \vdots \\ \beta_M(\mu_M)=0}}$$

…scalar products (off-shell):

$$\left\langle\widetilde{\Phi}_M(\mu_1,\mu_2,\ldots,\mu_M),\ \widetilde{\Phi}_M(\nu_1,\nu_2,\ldots,\nu_M)\right\rangle = 4^M \sum_{\sigma\in\mathcal{S}_M} \det\mathcal{M}^\sigma$$

$$\mathcal{M}^\sigma_{jj} = -\frac{\mu_j\,\rho(\mu_j) - \nu_{\sigma(j)}\,\rho(\nu_{\sigma(j)})}{\mu_j^2 - \nu_{\sigma(j)}^2} - \sum_{k\neq j}\frac{\mu_k^2 + \nu_{\sigma(k)}^2}{(\mu_j^2-\mu_k^2)(\nu_{\sigma(j)}^2 - \nu_{\sigma(k)}^2)}\,,$$

$$\mathcal{M}^\sigma_{jk} = -\frac{\mu_k^2 + \nu_{\sigma(k)}^2}{(\mu_j^2-\mu_k^2)(\nu_{\sigma(j)}^2 - \nu_{\sigma(k)}^2)}\,, \quad \text{for} \quad j,k = 1,2,\ldots,M\,.$$

To summarize…

- We solved ABA for rational so(3) Gaudin model with **fully general** nontrivial boundary conditions (thanks to nontrivial choice of vacuum)
- Found solutions for Knizhnik-Zamolodchikov equations
- Presented closed-form formulas for (on-shell) norms and (off-shell) scalar products of Bethe vectors.

Thank you.