Dynamical supersymmetry of the Landau levels $\mathfrak{Dirac} = \mathfrak{Di} \oplus \mathfrak{Rac}$

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Outline

- Tits-Kantor-Koecher (TKK) Construction $\mathfrak{co}(\mathfrak{J})$

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- 3D Hatom Spectrum Generating Alg $SO(2,4)$, Barut, Bornzin, MICZ-Kepler

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Talk is based on the articles:

- Tekin Dereli, Philippe Nounahon, Todor Popov
  ”A Remarkable Dynamical Symmetry of
  the Landau Problem.”
  IoP Tekin Dereli’s Festschrift, Istanbul 2021
- Tekin Dereli, Philippe Nounahon, Todor Popov
  ”Landau Levels versus Hydrogen Atom”,
  Universe, **10**, 172, 2024.
- Philippe Nounahon and Todor Popov,
  ”Landau levels for the Haldane’s spheres”
  to appear in Proceedings of XV. International Workshop
  ”Lie Theory and Its Applications in Physics”, Varna 2023.

Tits-Kantor-Koecher construction for Jordan algebra $\mathfrak{J}$

$$\begin{array}{ccccc} Co(\mathfrak{J}_2^{\mathbb{C}}) & = & SO(2,4) & \xleftarrow{Kustaanheimo-Stiefel} & SU(2,2) \\ \uparrow x=x^t & & & & \uparrow \psi=\psi^c \\ Co(\mathfrak{J}_2^{\mathbb{R}}) & = & SO(2,3) & \xleftarrow{Levi-Civita} & Sp(4,\mathbb{R}) \end{array}$$

Oscillator Realization $\mathfrak{co}(\mathfrak{J})$ (Murat Günaydin)

Compactified Minkowski space $\mathcal{M}^{1,3} := S^1 \times S^3/\mathbb{Z}_2$

$\mathbb{R}^{1,3}$ with metric $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$ as a spinor:

$$\tilde{x} = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \quad (\boldsymbol{x})^2 = \det \tilde{x} = \eta_{\mu\nu} x^\mu x^\nu \ .$$

Cayley transform C maps a Hermitian matrix to a unitary

$$C : H_2(\mathbb{C}) \to U(2) \ , \qquad U = \frac{1 - i\tilde{x}}{1 + i\tilde{x}} \ .$$

Compactified Minkowski space $\mathcal{M}^{1,3}$

$$C : \mathbb{R}^{1,3} \to \mathcal{M}^{1,3} := U(2) = U(1) \times SU(2)/\mathbb{Z}_2$$

is a homogeneous space of $SU(2,2)$: projective quadric in $\mathbb{R}^{2,4}$

$$\mathcal{M}^{1,3} = Q/\mathbb{R}^* \ , \quad Q = \{y_{-1}^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = \eta_{AB} y^A y^B = 0\} \ .$$

Kustaanheimo–Stiefel (*KS*) transform

$$KS: \quad T^+S^3 \to T^+S^2 \quad \subset (\mathbb{R}^*)^4 \times \mathbb{R}^4 \to (\mathbb{R}^*)^3 \times \mathbb{R}^3$$

$$\begin{array}{lllllll} q_1 & = & u_1u_3 + u_2u_4 \,, & p_1 & = & -(u_1w_3 + w_1u_3 + u_2w_4 + w_2u_4)/|\mathbf{z}|^2 \,, \\ q_2 & = & u_2u_3 - u_1u_4 \,, & p_2 & = & -(u_2w_3 + w_2u_3 - w_1u_4 - u_1w_4)/|\mathbf{z}|^2 \,, \\ q_3 & = & -u_1^2 - u_2^2 + u_3^2 + u_4^2 \,, & p_3 & = & (u_1w_1 + u_2w_2 - u_3w_3 - u_4w_4)/|\mathbf{z}|^2 \,. \end{array}$$

The coordinates on T^+S^3 are subject to the constraint

$$K = u_1w_2 - u_2w_1 + u_3w_4 - u_4w_3 = 2s \,.$$

a vector $\boldsymbol{x} \in \mathbb{R}^3$ is a "square root" of a spinor $\boldsymbol{Z} \in \mathbb{C}^2$

$$|\boldsymbol{q}| = |\boldsymbol{Z}|^2 \,, \qquad \boldsymbol{Z} = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} \,, \qquad |\boldsymbol{Z}|^2 = \boldsymbol{Z}^\dagger \boldsymbol{Z}. \qquad (1)$$

The Hopf fibration and regularized 4D MICZ-Kepler motion

$$0 \to S^1 \hookrightarrow S^3 \to S^2 \to 0 \,, \qquad J(\boldsymbol{Z}, \boldsymbol{W}) = \frac{1}{2}|\boldsymbol{Z}|^2 + \frac{1}{2}|\boldsymbol{W}|^2$$

Kustaanheimo-Stiefel spinorial regularization

Kustaanheimo-Stiefel transform

$$-x_k = \mathbf{z}\sigma_k\bar{\mathbf{z}}\ , \qquad \gamma = \arg z_1 + \arg z_2\ .$$

$$S^3_u := \left\{ \mathbf{z} := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 : \mathbf{z}^\dagger \mathbf{z} = \bar{z}_1 z_1 + \bar{z}_2 z_2 = u \right\}\ .$$

Dynamical group of H-atom/Charge-Dyon system (Barut & co)

3D H-atom	$\leftrightarrow$	4D Harmonic Oscillator
$SO(2,4)$	$\cong$	$SU(2,2)$ (Haldane's sphere)
2D H-atom	$\leftrightarrow$	2D Harmonic Oscillator
$SO(2,3)$	$\cong$	$Sp(4,\mathbb{R})$(planar Landau)

$SU(2,2)$ massless UIR of helicity s (Mack & Todorov)

$$J^{AB} = \bar{\psi}\sigma^{AB}\psi\ , \qquad C_1 = \bar{\psi}\psi \qquad SU(2,2)/\mathbb{Z}_2 \cong SO_0(2,4)$$

Newton-Hooke duality

$Sp(4, R)$ is the double-covering of the anti-de Sitter group

$$Sp(4, \mathbb{R})/\mathbb{Z}_2 \cong SO_0(2, 3) \; .$$

Newton–Hooke duality between
the harmonic oscillator (Landau model)

$$\begin{array}{ccc} \mathfrak{Dirac} := \mathfrak{Di} & \oplus & \mathfrak{Rac} \\ \{\text{2D charge-dyon system}\} & \oplus & \{\text{2D hydrogen atom}\} \end{array}$$

and quantum Coulomb–Kepler models (with magnetic charge)
$\mathfrak{Di}$ and $\mathfrak{Rac}$ Flato and Fronsdal Singletons

Landau levels/Harmonic oscillator

Hamiltonian for an Electron in a constant magentic field

$$H = \frac{1}{2\mu}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2 \ .$$

Oscillator representation

$$H = \frac{\hbar\omega}{2}\{a^+, a^-\}\ , \qquad [H, a^\pm] = \pm a^\pm\ ,$$

Landau levels and magnetic translations

$$H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle \qquad [H, b^\pm] = 0\ .$$

"zero modes" of the Hamiltonian H, and $[H, b^\pm] = 0$

Angular momentum operator:

$$L_z = \frac{\hbar}{2}(b^+b^- - a^+a^-)\ , \qquad [L_z, b^\pm] = \pm b^\pm\ , \qquad [L_z, H] = 0\ .$$

Landau levels/Harmonic oscillator

$$H = H_{osc} - \Omega L_z = \frac{p_x^2 + p_y^2}{2\mu} + \frac{\mu\Omega^2}{2}(x^2+y^2) - \Omega L_z \qquad \Omega = \omega/2\,.$$

cyclotron frequency $\omega = \frac{eB}{\mu c}$ and magnetic length $\ell^2 = \frac{\hbar}{\mu\omega} = \frac{\hbar c}{eB}$.

$$\begin{aligned} z &= (x+iy)/2\ell & \qquad \partial &= \frac{\partial}{\partial z} = \ell\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\,, \\ \bar{z} &= (x-iy)/2\ell & \qquad \bar{\partial} &= \frac{\partial}{\partial \bar{z}} = \ell\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\,. \end{aligned}$$

One Hilbert space, two different hamiltonians

$$\begin{aligned} H_{osc} &= \frac{\hbar\omega}{2}(z\bar{z} - \partial\bar{\partial}) = \frac{\hbar\omega}{4}\left(\{b^-, b^+\} + \{a^-, a^+\}\right)\,, \\ H &= \frac{\hbar\omega}{2}\{a^+, a^-\} = \frac{\hbar\omega}{2}(z\bar{z} - \partial\bar{\partial}) - \frac{\hbar\omega}{2}(z\partial - \bar{z}\bar{\partial})\,. \end{aligned}$$

$[H_{osc}, L_z] = 0$

Dirac remarkable $SO(2,3)$-representation

Eigenstates of compact $SO(2) \times SO(3) \subset SO(2,3)$

$$H\psi_{n,m} = \hbar\omega\left(n + \frac{1}{2}\right)\psi_{n,m}\,, \quad L_z\psi_{n,m} = \hbar m\psi_{n,m}\,, \quad n = n_r + \frac{|m| - m}{2}\,.$$

Lowest Landau Level

$$\psi_{0,m}(z,\bar{z}) = (m!)^{-\frac{1}{2}}(b^+)^m\psi_{0,0}(z,\bar{z}) \propto z^m e^{-z\bar{z}}\,,$$

Hilbert space is a sum of Landau levels and $SO(2,3)$-rep

$$\mathfrak{Dirac} = \bigoplus_{n\geq 0}\mathfrak{Dirac}_n = \bigoplus_{n\geq 0}\ \bigoplus_{m\geq -n}\psi_{n,m}(z,\bar{z})\,. \qquad (2)$$

$$\psi_{n,m}(z,\bar{z}) = \frac{(a^+)^n}{\sqrt{n!}}\frac{(b^+)^{n+m}\psi_{0,0}}{\sqrt{(n+m)!}} \propto e^{-|z|^2}\begin{cases} z^m L_n^{|m|}(2|z|^2) & m \geq 0 \\ \bar{z}^{|m|} L_{n+m}^{|m|}(2|z|^2) & -n \leq m < 0 \end{cases}.$$

$$\psi_{n,m}(z,\bar{z}) = C_{n,m}|z|^{|m|}L_{n_r}^{|m|}(2|z|^2)e^{-|z|^2}e^{im\varphi} \qquad n \geq 0 \quad m \geq -n\,.$$

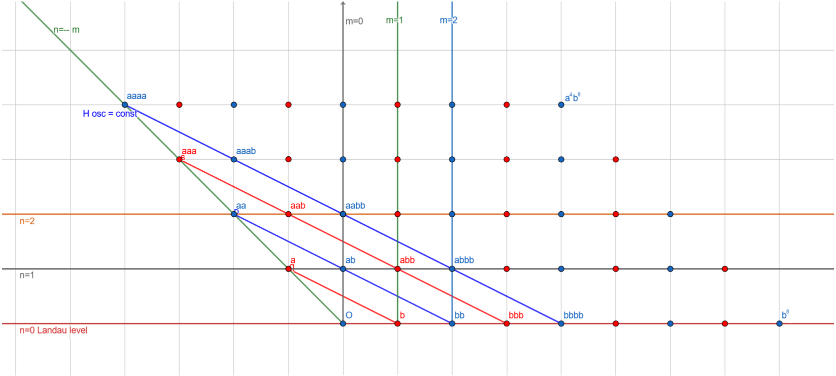

Figure: Hilbert space $\mathfrak{Dirac} = \mathfrak{Di} \oplus \mathfrak{Rac}$ has two $SO(2,3)$-orbits

Pseudo-vacuum of Landau level n

$$\psi_{n,-n} \propto \bar{z}^n e^{-z\bar{z}} \qquad \mathfrak{Dirac}_n = \bigoplus_{m \geq 0} \mathbb{C}(b^+)^m \psi_{n,-n} \; .$$

Levi-Civita transform

$$H_{osc} = \frac{\hbar\omega}{2}\left(a^+a^- + b^+b^- + 2\right) , \quad H = \hbar\omega\left(a^+a^- + \frac{1}{2}\right) = H_{osc} - \Omega L_z$$

$$H_{osc}\psi_{n,m}(z,\bar{z}) = \hbar\omega\left(N + \frac{1}{2}\right)\psi_{n,m}(z,\bar{z}) \qquad N = n + \frac{m}{2} = n_r + \frac{|m|}{2} .$$

Parity operator

$$\Pi\psi_{n,m}(z,\bar{z}) = \psi_{n,m}(e^{i\pi}z, e^{-i\pi}\bar{z}) = (-1)^m\psi_{n,m}(z,\bar{z}) .$$

two-sheeted covering the complex mapping

$$w = z^2 \qquad \text{where} \qquad z = |z|e^{i\varphi} \quad w = re^{i\theta}$$

mapping between $\varphi \in [0, 2\pi)$ and $\theta \in [0, 4\pi)$,
where θ winds twice for one period of φ:

$$r = z\bar{z} , \qquad \theta = 2\varphi .$$

2D hydrogen atom & electron-vortex system

Angular momentum J_z with half-integer spectrum

$$L_z = -i\hbar\frac{\partial}{\partial\varphi}\ , \qquad 2J_z = 2\times\left(-i\hbar\frac{\partial}{\partial\theta}\right)\ , \qquad m = 2j = 2(l+s)\ .$$

Energy spectrum of (magnetized) 2D Hatom

$$E_N = -\frac{\mu c^2\alpha^2}{2(N+\frac{1}{2})^2} = -\frac{\mu c^2\alpha^2}{2(n_r+|l+s|+\frac{1}{2})^2}\ .$$

with degeneracy $deg(E_N) = 2(N-|s|)+1$.
2D hydrogen atom with energy levels $N = |s|, |s|+1, \ldots$

$$\psi_{n,m}(r,\theta) = C^{(s)}_{N,j} e^{-\beta r}(\beta r)^{|j|} L^{2|j|}_{N-|j|}(2\beta r) e^{ij\theta} \qquad \beta = 1/\left(N+1/2\right)\ .$$

sections of $U(1)$-bundle

Boundary conditions

$$\Pi\psi_{n,m}(r,\theta) = \psi_{n,m}(r,\theta+2\pi) = (-1)^m\psi_{n,m}(r,\theta) \ ,$$

$$\psi_{n,m}(r,\theta) = e^{is\theta}\tilde{\psi}_{n,m}(r,\theta) \ , \qquad \tilde{\psi}_{n,m}(r,\theta+2\pi) = \tilde{\psi}_{n,m}(r,\theta) \ .$$

2D charge–vortex system $s = \frac{1}{2}$

$$H_{flux} = \frac{1}{2\mu}\left(\boldsymbol{p} - \frac{e}{c}\mathcal{A}\right)^2 - \frac{e^2}{r} \ , \quad \mathcal{A}(x,y) = \frac{g}{r^2}(-y,x) \ , \quad g = \hbar cs/e \, .$$

$$\boldsymbol{B}(\boldsymbol{r}) = rot\mathcal{A}(\boldsymbol{r}) = 2\pi g\delta(\boldsymbol{r}) \ .$$

(Hydrogen atom $s = 0$)

isotropic rays in $\mathbb{R}^{2,3}$ carry a linear $SO(2,3)$-representation

$$m_{ab} = \begin{pmatrix} 0 & m_{-10} & m_{-11} & m_{-12} & m_{-13} \\ & 0 & m_{01} & m_{02} & m_{03} \\ & & 0 & m_{12} & m_{13} \\ & & & 0 & m_{23} \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & h_\tau & M_1 & M_2 & D \\ & 0 & \Gamma_1 & \Gamma_2 & \tilde{h}_\rho \\ & & 0 & L_z & A_1 \\ & & & 0 & A_2 \\ & & & & 0 \end{pmatrix}.$$

SGA $\mathfrak{so}(2,3) \cong \mathfrak{sp}(4,\mathbb{R})$

Energy	hamiltonian	symmetry	integrals	inv. geometry
$E < 0$	$h_\tau \in \mathfrak{so}(2)$	$\mathfrak{so}(3)$	$L_z, \boldsymbol{A}$	$S^1 \times S^2$
$E > 0$	$\tilde{h}_\rho \in \mathfrak{so}(1,1)$	$\mathfrak{so}(1,2)$	$L_z, \boldsymbol{M}$	$H^1 \times H^2$
$E = 0$	$(h_\tau + \tilde{h}_\rho) \in \mathbb{R}$	$\mathfrak{so}(2) + \mathbb{R}^2$	$L_z, \boldsymbol{r}$	lightcone $*$
	$D \in \mathbb{R}$	$\mathfrak{so}(1,2)$	$L_z, \boldsymbol{\Gamma}$	AdS_4

The skewsymmetric $m_{ab} = -m_{ba}$, $a, b \in \{-1, 0, 1, 2, 3\}$

$$[m_{ab}, m_{cd}] = 0\,, \quad [m_{ab}, m_{bc}] = -i\eta_{bb} m_{ac}\,, \quad \eta_{ab} = diag(+,+,-,-,-)$$

$$\begin{aligned}
m_{12} &= \tfrac{1}{2}(z\partial - \bar{z}\bar{\partial}) &&= \tfrac{1}{4}(\{b^-, b^+\} - \{a^-, a^+\})\\
m_{23} &= \tfrac{1}{4}(z^2 + \bar{z}^2 - \partial^2 - \bar{\partial}^2) &&= \tfrac{1}{4}(\{a^-, b^+\} + \{a^+, b^-\})\\
m_{31} &= \tfrac{i}{4}(z^2 - \bar{z}^2 + \partial^2 - \bar{\partial}^2) &&= \tfrac{i}{4}(\{a^-, b^+\} - \{a^+, b^-\})\\
m_{1-1} &= \tfrac{1}{4i}(z^2 - \bar{z}^2 - \partial^2 + \bar{\partial}^2) &&= \tfrac{i}{4}\left(a^{+2} - a^{-2} + b^{-2} - b^{+2}\right)\\
m_{2-1} &= \tfrac{1}{4}(z^2 + \bar{z}^2 + \partial^2 + \bar{\partial}^2) &&= \tfrac{1}{4}\left(a^{-2} + a^{+2} + b^{-2} + b^{+2}\right)\\
m_{3-1} &= -\tfrac{i}{2}\left(z\partial + \bar{z}\bar{\partial} - 1\right) &&= -\tfrac{i}{4}(\{a^-, b^-\} - \{a^+, b^+\})\\
m_{01} &= -\tfrac{1}{2}\left(z\bar{\partial} - \bar{z}\partial\right) &&= -\tfrac{1}{4}\left(a^{-2} + a^{+2} - b^{-2} - b^{+2}\right)\\
m_{02} &= -\tfrac{i}{2}\left(\bar{z}\partial + z\bar{\partial}\right) &&= -\tfrac{i}{4}\left(a^{-2} - a^{+2} + b^{-2} - b^{+2}\right)\\
m_{03} &= \tfrac{1}{2}\left(z\bar{z} + \partial\bar{\partial}\right) &&= -\tfrac{1}{4}(\{a^+, b^+\} + \{a^-, b^-\})\\
m_{-10} &= \tfrac{1}{2}\left(z\bar{z} - \partial\bar{\partial}\right) &&= \tfrac{1}{4}(\{a^-, a^+\} + \{b^-, b^+\})
\end{aligned}$$

Cayley transform and Klein-Dirac quadric

$$y_{-1}^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 = R^2 = \eta_{ab} y^a y^b \,, \qquad \eta_{ab} = diag(+,+,-,-,-) \,.$$

The symplectic Cayley transform C maps the real symmetric matrix $S = S^T$ to the real symplectic matrix

$$C : H_2(\mathbb{R}) \to Sp(2,\mathbb{R}) \,, \quad C(S) = M := \frac{\mathbb{1} - \mathcal{J} S}{\mathbb{1} + \mathcal{J} S} \,, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

compactifies the flat Minkowski space $\mathbb{R}^{1,2}$ to the compactified Minkowski space:

$$C : \mathbb{R}^{1,2} \to \mathcal{M}^{1,2} := Sp(2,\mathbb{R}) \cong (S^1 \times S^2)/\mathbb{Z}_2 \,.$$

Weyl Spinors and $Sp(4,\mathbb{R})/\mathbb{Z}_2 \cong SO(2,3)$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} + \partial \\ z + \bar{\partial} \end{pmatrix} , \quad \chi^*_\alpha = (b^+ \ \ a^+) .$$

2D H-atom $\mathfrak{so}(2,3)$-spinorial representation, Barut & Duru

$$\begin{aligned} m_{ij} &= \tfrac{1}{2}\epsilon_{ijk}\chi^*\sigma_k^T\chi , & m_{-1i} &= \tfrac{i}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T - \chi^T\epsilon\sigma_i^T\chi) , \\ m_{-10} &= \tfrac{1}{2}(\chi^*\chi + 1) , & m_{0i} &= \tfrac{1}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T + \chi^T\epsilon\sigma_i^T\chi) \end{aligned}$$

4D Dirac spinor yields 3D H-atom $\mathfrak{so}(2,4)$-spinorial rep

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta^\alpha_\beta , \qquad [\psi^\alpha, \psi^\beta] = 0 .$$

via Mack & Todorov ladder $U(2,2)$ representation

$$J^{AB} = \bar{\psi}\sigma^{AB}\psi , \qquad C_1 = \bar{\psi}\psi$$

its Majorana reduction is $Sp(4,\mathbb{R})$-rep: Stoyanov & Todorov

$$\psi^c = \psi$$

Dynamical group $SO(2,4)$ of MICZ-Kepler model

$$\begin{pmatrix} 0 & L_{-10} & L_{-11} & L_{-12} & L_{-13} & L_{-15} \\ & 0 & L_{01} & L_{02} & L_{03} & L_{05} \\ & & 0 & L_{12} & L_{13} & L_{15} \\ & & & 0 & L_{23} & L_{25} \\ & & & & 0 & L_{35} \\ & & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_\tau & M_1 & M_2 & M_3 & D \\ & 0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & \tilde{H}_\rho \\ & & 0 & J_3 & -J_2 & A_1 \\ & & & 0 & J_1 & A_2 \\ & & & & 0 & A_3 \\ & & & & & 0 \end{pmatrix}.$$

MICZ-Kepler SGA $\mathfrak{so}(2,4)$ with helicity s

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{r}\times\boldsymbol{\pi} - s\hat{\boldsymbol{r}} & \boldsymbol{\Gamma} = r\boldsymbol{\pi}, \qquad D &= \boldsymbol{r}\cdot\boldsymbol{\pi} - i\,, \\ \boldsymbol{A} &= \tfrac{1}{2}\boldsymbol{r}\pi^2 - \boldsymbol{\pi}(\boldsymbol{r}\cdot\boldsymbol{\pi}) + \tfrac{s}{r}\boldsymbol{J} + \tfrac{s^2}{2r^2}\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}\,, & H_\tau &= \tfrac{1}{2}\left(r\pi^2 + \tfrac{s^2}{r} + r\right), \\ \boldsymbol{M} &= \tfrac{1}{2}\boldsymbol{r}\pi^2 - \boldsymbol{\pi}(\boldsymbol{r}\cdot\boldsymbol{\pi}) + \tfrac{s}{r}\boldsymbol{J} + \tfrac{s^2}{2r^2}\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}\,, & \tilde{H}_\rho &= \tfrac{1}{2}\left(r\pi^2 + \tfrac{s^2}{r} - r\right). \end{aligned}$$

Energy	hamiltonian	symmetry	integrals	invariant geometry
$E<0$	$H_\tau \in \mathfrak{so}(2)$	$\mathfrak{so}(4)$	$\boldsymbol{J},\boldsymbol{A}$	$S^1\times S^3$
$E>0$	$\tilde{H}_\rho \in \mathfrak{so}(1,1)$	$\mathfrak{so}(1,3)$	$\boldsymbol{J},\boldsymbol{M}$	$H^1\times H^3$
$E=0$	$(H_\tau+\tilde{H}_\rho)\in\mathbb{R}$	$\mathfrak{so}(3)+\mathbb{R}^3$	$\boldsymbol{J},\boldsymbol{r}$	lightcone $*$
	$D\in\mathbb{R}$	$\mathfrak{so}(1,3)$	$\boldsymbol{J},\boldsymbol{\Gamma}$	AdS_5

Parabosons and Sigletons

n parabosons close $\mathfrak{osp}(1|2n)$, Ganchev, Palev

$$[\{b_i^+, b_j^-\}, b_k^+] = 2\delta_{jk} b_i^+ \qquad [\{b_i^+, b_j^+\}, b_k^+] = 0$$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} \qquad \chi^*_\alpha = (b_1^+ \;\; b_2^+)$$

$n = 2$ parabosons $\mathfrak{osp}(1|4)$ Landau problem

$$\mathfrak{osp}(1|4)_0 = \mathfrak{sp}(4, \mathbb{R})$$

unitary massless conformal $\mathfrak{so}(2,3)$-represenations $D(E_0, s)$
$m_{-10} = H_{osc}$ minimal *conformal energy* E_0
$m_{12} = L_z$ minimal angular momentum, *helicity* s
Fronsdal, Flato Singeltons
"Massless particles, conformal group, and de Sitter universe"

SGA $\mathfrak{osp}(1|4)$ of the Landau levels

unitary massless conformal $\mathfrak{so}(2,3)$-represenations $D(E_0, s)$

Lemma

The Hilbert space $\mathfrak{Dirac}$ is a reducible $\mathfrak{so}(2,3)$-representation:

$$\mathfrak{Dirac} := \mathfrak{Di} \oplus \mathfrak{Rac}$$

The states $\psi_{n,m}(z,\bar{z}) \in \mathfrak{Dirac}$ belong to the orbit of a vacuum or a pseudo-vacuum

$$\psi_{0,0} \in \mathfrak{Rac} = D(1/2, 0) \quad \textit{and} \quad \psi_{0,1} = b^{+}\psi_{0,0} \in \mathfrak{Di} = D(1, 1/2) \, .$$

Theorem (SGA of Landau levels)

The superalgebra $\mathfrak{osp}(1|4)$ is SGA for the Landau levels space $\mathfrak{Dirac}$. It generates all the Landau level states

$$\mathfrak{Dirac} \cong \mathfrak{osp}(1|4)\psi_{0,0} \, .$$

Conformal Landscape for Landau problem

Kepler model	Oscillator	MICZ-Kepler model
H atom in $\mathbb{R}^3$ helicity $s = 0$	Oscillator in $\mathbb{R}^4$ Landau model on S^2	charge-dyon in $\mathbb{R}^3$ (magnetized H-atom)
even Landau states $Rac = D(E_0 = \frac{1}{2}, s = 0)$	Planar Landau model $Dirac = Rac \oplus Di$	odd Landau states $Di = D(E_0 = 1, s = \frac{1}{2})$
H atom in $\mathbb{R}^2$	Oscillator in $\mathbb{R}^2$	charge-vortex in $\mathbb{R}^2$

$$Co(\mathfrak{J}_2^{\mathbb{C}}) \quad = \quad SU(2,2) \qquad \text{Haldane's sphere}$$

$$x = x^t \qquad\qquad \psi = \psi^c$$

$$Co(\mathfrak{J}_2^{\mathbb{R}}) \quad = \quad Sp(4,\mathbb{R}) \qquad \text{planar Landau model}$$

Euclidean Jordan algebra $\mathfrak{J}$

Algebra of observables in Quantum Mechanics
(Pascual Jordan)

$$x \circ y = y \circ x = \frac{1}{2}(xy + yx)$$

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x) \qquad \forall x, y \in \mathfrak{J} \, .$$

$\mathfrak{J}$ is commutative but is not associative
Euclidean Jordan algebra (Jordan, von Neumann, Wigner)

$$x^2 + y^2 = 0 \qquad \Rightarrow x = 0 \quad \text{and} \quad y = 0$$

it is power-associativity, $x^{m+n} = x^m \circ x^n$

Tits-Kantor-Koecher Construction

conformal (Möbius) algebra

$$\mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \cong \mathfrak{J} \oplus \mathfrak{str}(\mathfrak{J}) \oplus \mathfrak{J}^*$$

$\mathfrak{co}(\mathfrak{J})$ is a 3-graded Lie algebra with involution $\dagger$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \qquad \mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}$$

$$(x, y, z) := [[x, y^\dagger], z] \qquad x, y, z \in \mathfrak{g}_{+1} \cong \mathfrak{J} \,.$$

Jordan algebra and Jordan triple product

$$(abc) = a \circ (b \circ c) - b \circ (a \circ c) + (a \circ b) \circ c \, .$$

Jordan Triple System

$$\begin{aligned} (abc) &= (cba) \qquad (3) \\ (ab(cdx)) - (cd(abx)) &= (a(dcb)x) - ((cda)bx) \, . \end{aligned}$$

a linear map $S_x^y : \mathfrak{J} \to \mathfrak{J}$ through

$$S_x^y(z) = (xyz) \, .$$

Structure algebra $\mathfrak{str}(\mathfrak{J})$

$$[S_a^b, S_c^d] = S_{(abc)}^d - S_c^{(dab)} = S_a^{(bcd)} - S_{(cda)}^b \qquad (4)$$

TKK conformal algebra $\mathfrak{co}(\mathfrak{J})$

Structure constants $\Sigma_{\mu\rho}^{\nu\sigma}$

$$S_x^y(z) = (xyz) \qquad (e_\mu, e_\nu, e_\rho) = \Sigma_{\mu\rho}^{\nu\sigma} e_\sigma \ .$$

determine conformal algebra $\mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \cong \mathfrak{J}$

$$\begin{aligned}
[U_a, U^b] &= -2S_a^b \,, & [U_a, U_b] &= 0 \,, & U_a &\in \mathfrak{g}_{-1} \\
[S_a^b, U_c] &= U_{(abc)} \,, & [S_a^b, S_c^d] &= S_{(abc)}^d - S_c^{(bad)} \,, & S_a^b &\in \mathfrak{g}_0 \\
[S_a^b, U^c] &= -U^{(bac)} \,, & [U^a, U^b] &= 0 \,, & U^b &\in \mathfrak{g}_{+1}
\end{aligned}$$

Guowu Meng $\mathfrak{J}$-Kepler problem

Jordan Triple Structure constants

$$x = x^{\mu}\sigma_{\mu} = x = x^{\mu}\sigma_{\mu} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \qquad x \in \mathfrak{J}_2^{\mathbb{C}}$$

$$(\sigma_\alpha, \sigma_\beta, \sigma_\gamma) = \Sigma_{\alpha\gamma}^{\beta\rho}\sigma_\rho = \sigma_\alpha \circ (\sigma_\beta \circ \sigma_\gamma) - \sigma_\beta \circ (\sigma_\gamma \circ \sigma_\alpha) + (\sigma_\alpha \circ \sigma_\beta) \circ \sigma_\gamma$$

gives back the concise formula

$$\Sigma_{\alpha\gamma}^{\beta\rho} = \delta_\gamma^\rho \delta_\alpha^\beta + \delta_\alpha^\rho \delta_\gamma^\beta - g^{\beta\rho} g_{\alpha\gamma} \ . \tag{5}$$

Günaydın's Oscillator Realization of $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4)$

$\mathfrak{co}(\mathfrak{J})$	operator	$\in \mathfrak{co}(\mathfrak{J})$	mapping	x−rep basis $\mathfrak{so}(2,4)$	deg(x)
$\mathfrak{J}$	$U_a = -ia^\mu P_\mu$	$\in \mathfrak{g}_{-1}$	$x \mapsto a$	$P_\nu = i\partial_\nu$	0
$\mathfrak{str}(\mathfrak{J})$	$S_a^b = ia^\nu b_\mu S_\nu^\mu$	$\in \mathfrak{g}_0$	$x \mapsto (a,b,x)$	$S_\nu^\mu = -i\Sigma_{\nu\alpha}^{\mu\beta} x^\alpha \partial_\beta$	1
$\mathfrak{J}^*$	$U^b = ib_\mu K^\mu$	$\in \mathfrak{g}_{+1}$	$x \mapsto -(x,b,x)$	$K^\mu = i\Sigma_{\nu\alpha}^{\mu\beta} x^\nu x^\alpha \partial_\beta$	2

Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$ and 3-graded Lie algebra $\mathfrak{so}(2,4)$

$\mathfrak{g}_{-1}$	$-iP_\nu$	$=$	∂_ν	translations
$\mathfrak{g}_0$	$iM^\mu{}_\nu$	$=$	$-x^\mu\partial_\nu + x_\nu\partial^\mu$	Lorentz transformations
$\mathfrak{g}_0$	iD	$=$	$x^\mu\partial_\mu$	dilatation
$\mathfrak{g}_{+1}$	iK^μ	$=$	$-2x^\mu x^\nu\partial_\nu + x^\nu x_\nu\partial^\mu$	special conformal

Conformal algebra of Minkowski spacetime $\mathbb{R}^{1,3}$

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4) \text{ acting on } \mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^* \cong (S^1 \times S^3)/\mathbb{Z}_2$$

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4) = \underbrace{(\mathfrak{J}_2^{\mathbb{C}})^*}_{K^\mu} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,3)}_{M^\mu_\nu} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{C}}}_{P_\nu} \qquad \mu, \nu = 0, 1, 2, 3$$

$$x = x^\mu \sigma_\mu = x = x^\mu \sigma_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \qquad x \in \mathfrak{J}_2^{\mathbb{C}}$$

$$\Sigma^{\beta\rho}_{\alpha\gamma} = \delta^\rho_\gamma \delta^\beta_\alpha + \delta^\rho_\alpha \delta^\beta_\gamma - g^{\beta\rho} g_{\alpha\gamma} \, . \tag{6}$$

many layer realization of the Erlangen program

$$Aut(\mathfrak{J}) \subset Str(\mathfrak{J}) \subset Co(\mathfrak{J})$$

$$SO(3) \subset SO(1,3) \subset SO(2,4)$$

Group	Transformations		Space
$Aut(\mathfrak{J})$	rotations	$x' = Ox$	eulidean
$Str_0(\mathfrak{J})$	Lorentz group	$x' = \Lambda x$	pseudo-euclidean
$\mathcal{P}(\mathfrak{J})$	Poincaré group	$x' = \Lambda x + a$	affine pseudo-euclid
$\mathcal{P}_+(\mathfrak{J})$	Similitude group	$x' = e^\rho(\Lambda x + a)$	rescaled affine pseu
$Co(\mathfrak{J})$	Möbius group	$z' = \frac{az+b}{cz+d}$	projective

acting on projective space of rays in the Hilbert space

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4) = \underbrace{(\mathfrak{J}_2^{\mathbb{C}})^*}_{K^\mu} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,3)}_{M^\mu_\nu} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{C}}}_{P_\nu}$$

real Pauli matrices with coordinates $\{y_0, y_1, y_2\}$

$$y = \sum_{\mu=0,1,3} x^\mu \sigma_\mu = \begin{pmatrix} y_0 + y_2 & y_1 \\ y_1 & y_0 - y_2 \end{pmatrix}, \quad y^T = y . \tag{7}$$

conformal algebra of Minkowski space $\mathbb{R}^{1,2}$ acting on $\mathcal{M}_{1,2} = \mathcal{N}/\mathbb{R}^* \cong (S^1 \times S^2)/\mathbb{Z}_2$

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) = \mathfrak{so}(2,3) = \underbrace{(\mathfrak{J}_2^{\mathbb{R}})^*}_{K^{\tilde{\mu}}} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,2)}_{M^{\tilde{\mu}}_{\tilde{\nu}}} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{R}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{R}}}_{P_{\tilde{\nu}}} \qquad \tilde{\mu}, \tilde{\nu} = 0, 1, 2$$