Exact models of gravitational waves based on Shapovalov wave spacetimes

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Let us recall the main statements of the Stäckel spaces theory.

Definition

Let V_n be a n–dimensional Riemannian space with metric tensor g_{ij}. The Hamilton – Jacobi equation

$$g^{ij} S_{,i} S_{,j} = m^2 \qquad i, j = 1, ...n \tag{1}$$

can be integrated by **complete separation of variables method** *if co-ordinate set $\{u^i\}$ exists for which complete integral can be presented in the form:*

$$S = \sum_{i=1}^{n} \phi_i(u^i, \lambda_k) \tag{2}$$

where $\lambda_1 ... \lambda_n$ – is the essential parameter.
V_n is called the Stäckel space if the Hamilton–Jacobi equation (1) can be integrated by complete separation of variables method.

Theorem

Let V_n be the Stäckel space. Then g_{ij} in privileged co-ordinate set can be shown in the form

$$g^{ij} = (\Phi^{-1})^{\nu}_{n} G^{ij}_{\nu},$$

$$G^{ij}_{\nu} = G^{ij}_{\nu}(u^{\nu}), \qquad \Phi^{\nu}_{\mu} = \Phi^{\nu}_{\mu}(u^{\mu}) \tag{3}$$

$$G^{ij}_{\nu} = \delta^i_{\nu}\delta^j_{\nu}\varepsilon_{\nu}(u^{\nu}) + (\delta^i_{\nu}\delta^j_{p} + \delta^j_{\nu}\delta^i_{p})G^{\nu p}_{\nu}(u^{\nu}) + \delta^i_{p}\delta^j_{q}h^{pq}_{\nu}(u^{\nu}),$$

$$p, q = 1, ...N, \quad \nu, \mu = N+1, ..., n.$$

where $\Phi^{\nu}_{\mu}(u^{\mu})$ – is called the Stäckel matrix.

Geodesic equations of Stäckel spaces admit **the first integrals** that commutes pairwise with respect to the Poisson bracket

$$\underset{\mu}{X} = (\Phi^{-1})^{\nu}_{\mu}(\varepsilon_{\nu} p_{\nu}^2 + 2G_{\nu}^{\nu p} p_p p_{\nu} + h_{\nu}^{pq} p_p p_q),$$

$$\underset{p}{Y} = \underset{p}{Y}{}^{i} p_i, \tag{4}$$

$$p, q = 1, ...N; \quad \nu, \mu = N+1, ..., n.$$

$\Phi^{\nu}_{\mu}(u^{\mu})$ – is called the Stäckel matrix,
functions ε_{ν}, $G_{\nu}^{\nu p}$, h_{ν}^{pq} depends only from u^{ν}, p_i is momentum.

If we write the functions X_ν, Y_p in the form:

$$\underset{\nu}{X} = \underset{\nu}{X}{}^{ij} p_i p_j, \qquad \underset{p}{Y} = \underset{p}{Y}{}^{i} p_i \tag{5}$$

then

$$\underset{\nu}{X}{}_{(ij;k)} = \underset{p}{Y}{}_{(i;j)} = 0$$

(the semicolon denotes the covariant derivative and the brackets denote symmetrization).

Therefore $Y_p{}^i$, $X_\nu{}^{ij}$ are the components of **vector and tensor Killing fields** respectively.

Definition

Pairwise commuting Killing vectors $Y_p{}^i$, where $p = 1, ...N$ and Killing tensors $X_\nu{}^{ij}$, where $\nu = N+1, ...n$ form a complete set of the type $(N.N_0)$, where

$$N_0 = N - rank||\underset{p}{Y}{}^{i}\underset{q}{Y}{}_{i}||$$

Theorem

A necessary and sufficient geometrical criterion of a **Stäckel** *space is the presence of a complete set of Killing fields of the type* $(N.N_0)$*.*

Then the Hamilton-Jacobi equation can be integrated by the complete separation of variables method if and only if the complete set of the first integrals exists.

Definition

Space - time is called a **Stäckel** *one of the type* $(N.N_0)$ *if the complete set of the type* $(N.N_0)$ *exists.*

Definition

Stäckel spacetime whose metric in a privileged coordinate system depends on a null (wave) variable (along which the interval vanishes) is called a **Shapovalov** *wave spacetime.*

Part II. Pure radiation in spacetime models that admit integration of the eikonal equation by the separation of variables method

In this part of report we propose an approach to the modeling of spacetime with pure radiation based on approach with the following assumptions:

Assumption I

Realistic theory of gravity is a metric theory, i.e. gravity is modeled by a metric tensor, and test particles and radiation move along the geodesic lines of spacetime;

Assumption II

The law of conservation of energy-momentum of matter is satisfied;

Assumption III

To construct exact integrable models, we will use spaces that allow the integration of the eikonal equation by the method of separation of variables.

The energy-momentum tensor of pure radiation:

$$T_{ij} = \varepsilon\, L_i L_j, \tag{6}$$

where ε – energy density and L_i – wave vector of radiation.
The wave vector L_i is an isotropic vector and satisfies the norm condition:

$$g^{ij} L_i L_i = 0. \tag{7}$$

The energy-momentum conservation law:

$$\nabla^i T_{ij} = 0. \tag{8}$$

where ∇_i is the covariant derivative.
Below are listed the solutions for all types of this spacetimes without the use of field equations of the concrete theory of gravitation.

The eikonal equation

$$g^{ij}\nabla_i\Psi\nabla_j\Psi = 0. \tag{9}$$

Definition

Let V_n be a n–dimensional Riemannian space with metric tensor g_{ij}. The eikonal equation (9) can be integrated by **complete separation of variables method** if co-ordinate set $\{u^i\}$ exists for which complete integral can be presented in the "separated" form:

$$\Psi = \sum_{i=1}^{n} \psi_i(u^i, \lambda_1, ..., \lambda_n) \tag{10}$$

where $\lambda_1 ... \lambda_n$ – is the essential parameter.

Such spaces are called the **conformally Stäckel spaces**.

Conformally Stäckel spacetimes (3.1) type admits 3 commuting Killing vectors $Y^i_{(p)}$ $(p = 1, 3)$, but $rank\,|Y^i_{(p)} g_{ij} Y^j_{(q)}| = 2$. In a privileged coordinate system the metric of a conformally Stäckel spacetimes (3.1) type can be written in the following form, where the variable x^0 is a null ("wave") variable:

$$g^{ij} = \frac{1}{\Delta} \begin{pmatrix} 0 & 1 & a_0 & b_0 \\ 1 & 0 & 0 & 0 \\ a_0 & 0 & c_0 & f_0 \\ b_0 & 0 & f_0 & d_0 \end{pmatrix}, \qquad (11)$$

where $\Delta = \Delta(x^0, x^1, x^2, x^3)$ and a_0, b_0, c_0, d_0, f_0 are functions of a variable x^0.

The wave vector of the radiation has the form:

$$L_0 = L_0(x^0), \quad L_1 = \alpha, \quad L_2 = \beta, \quad L_3 = \gamma,$$
$$\alpha, \beta, \gamma - const.$$

From the normalization condition and the conservation equations we obtain:

$$L_i = \left(L_0(x^0), \alpha, \beta, \gamma\right), \quad \alpha, \beta, \gamma - const, \tag{12}$$

$$L_0 = \frac{-(\beta^2 c_0 + 2\beta\gamma f_0 + \gamma^2 d_0)}{2\,(\alpha + \beta a_0 + \gamma b_0)}. \tag{13}$$

For the energy density of radiation we obtain:

$$\varepsilon = F(X,Y,Z)\Delta\sqrt{-\det g^{ij}}/(\alpha + \beta a_0 + \gamma b_0), \tag{14}$$

$$\begin{aligned} X &= x^1 - \int \frac{L_0}{(\alpha + \beta a_0 + \gamma b_0)}\, dx^0, \\ Y &= x^2 - \int \frac{(a_0 L_0 + \beta c_0 + \gamma f_0)}{(\alpha + \beta a_0 + \gamma b_0)}\, dx^0, \\ Z &= x^3 - \int \frac{(b_0 L_0 + \beta f_0 + \gamma d_0)}{(\alpha + \beta a_0 + \gamma b_0)}\, dx^0, \end{aligned}$$

where $F(X,Y,Z)$ is an arbitrary function of its variables.

Conformally Stäckel spacetimes (2.1) type admits two commuting Killing vectors $Y^i_{(p)}$ $(p=1,2)$, but $rank\,|Y^i_{(p)}g_{ij}Y^j_{(q)}|=1$. In a privileged coordinate system the metric of a conformally Stäckel spacetimes (2.1) type can be written in the following form, where x^1 is a null ("wave") variable:

$$g^{ij}=\frac{1}{\Delta}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & f_1(x^1) & 1\\ 0 & f_1(x^1) & A & B\\ 0 & 1 & B & C\end{pmatrix}, \tag{15}$$

$$\Delta=\Delta(x^0,x^1,x^2,x^3),\quad A=a_0(x^0)+a_1(x^1),$$
$$B=b_0(x^0)+b_1(x^1),\quad C=c_0(x^0)+c_1(x^1).$$

The wave vector of radiation L_i has the form:

$$L_0=L_0(x^0),\quad L_1=L_1(x^1),$$
$$L_2=\alpha,\quad L_3=\beta,\quad \alpha,\beta,\gamma-const. \tag{16}$$

The wave vector of radiation has the form:

$$L_i = \big(L_0(x^0), L_1(x^1), \alpha, \beta\big)\,, \quad \alpha, \beta, \gamma - const,$$

$$L_0 = \sqrt{\gamma - \alpha^2 a_0 - 2\alpha\beta b_0 - \beta^2 c_0}, \tag{17}$$
$$L_1 = (-\gamma - \alpha^2 a_1 - 2\alpha\beta b_1 - \beta^2 c_1)/\big(2\,(\alpha f_1 + \beta)\big).$$

The radiation energy density has the form:

$$\varepsilon = \frac{F(X,Y,Z)\Delta\sqrt{-\det g^{ij}}}{L_0(\alpha f_1 + \beta)}, \tag{18}$$

$$X = \int \frac{dx^0}{L_0} - \int \frac{dx^1}{(\alpha f_1 + \beta)},$$

$$Y = x^2 - \int \frac{(\alpha a_0 + \beta b_0)}{L_0}\,dx^0 - \int \frac{(\alpha a_1 + \beta b_1 + f_1 L_1)}{\alpha f_1 + \beta}\,dx^1,$$

$$Z = x^3 - \int \frac{(\alpha b_0 + \beta c_0)}{L_0}\,dx^0 - \int \frac{(\alpha b_1 + \beta c_1 + L_1)}{\alpha f_1 + \beta}\,dx^1,$$

where $F(X,Y,Z)$ is an arbitrary function of its variables.

Conformally Stäckel spacetimes (1.1) type admits one Killing vector. In a privileged coordinate system the metric can be written in the following form, where x^1 is a null ("wave") variable:

$$g^{ij} = \frac{1}{\Delta} \begin{pmatrix} \Omega & V^1 & 0 & 0 \\ V^1 & 0 & 0 & 0 \\ 0 & 0 & V^2 & 0 \\ 0 & 0 & 0 & V^3 \end{pmatrix}, \tag{19}$$

$$\Delta = \Delta(x^0, x^1, x^2, x^3),$$
$$V^1 = t_2(x^2) - t_3(x^3), \quad V^2 = t_3(x^3) - t_1(x^1),$$
$$V^3 = t_1(x^1) - t_2(x^2), \quad \Omega = \omega_\mu(x^\mu)V^\mu, \quad \mu, \nu = 1...3.$$

The wave vector of radiation has the following "separated"form:

$$L_i = \Big(\alpha, L_1(x^1), L_2(x^2), L_3(x^3)\Big), \quad \alpha - const. \tag{20}$$

The wave vector of radiation has the form:

$$L_0 = \alpha, \quad L_1 = \frac{1}{2\alpha}(\beta t_1 - \alpha^2\omega_1 + \gamma), \quad \alpha, \beta, \gamma - const,$$

$$L_2 = \sqrt{\beta t_2 - \alpha^2\omega_2 + \gamma}, \quad L_3 = \sqrt{\beta t_3 - \alpha^2\omega_3 + \gamma}. \tag{21}$$

The energy density of the radiation has the form:

$$\varepsilon = F(X, Y, Z)\,\Delta\sqrt{-\det g^{ij}}/(L_2\,L_3), \tag{22}$$

$$X = x^0 - \frac{1}{\alpha}\int (L_1 + \alpha\omega_1)\,dx^1 - \alpha\left(\int \frac{\omega_2}{L_2}\,dx^2 + \int \frac{\omega_3}{L_3}\,dx^3\right),$$

$$Y = -\frac{1}{\alpha}\int t_1\,dx^1 + \int \frac{t_2}{L_2}\,dx^2 + \int \frac{t_3}{L_3}\,dx^3,$$

$$Z = \frac{x^1}{\alpha} + \int \frac{dx^2}{L_2} + \int \frac{dx^3}{L_3},$$

where $F(X, Y, Z)$ is an arbitrary function of its variables.

Conclusion on Part II: Pure radiation in spacetime models that admit integration of the eikonal equation by the separation of variables method

I. For spacetime models with pure radiation we suggest a method for obtaining analytical solutions in **any metric theories of gravity** based on the use of coordinate systems that admit separation of variables in the eikonal equation.

II. The method is based on **integrating the energy-momentum conservation equations**.

III. In the report we present **a classification of the solutions** of the energy-momentum conservation equations for all types of spacetimes that allow the separation of variables in the eikonal equation.

Part III. Shapovalov gravitational wave exact solutions for spatially homogeneous models of Bianchi universes

Let us formulate the problem of constructing models of a Bianchi universes that allow gravitational wave exact solutions.
We assume that:

- the metric of spacetime is a plane-wave metric, that is, there exists a coordinate system where the metric depends only on one null wave variable;
- the spacetime under consideration is spatially homogeneous, that is, there exists a subgroup of spacetime isometries with 3-dimensional space-like orbits;
- the metric of spacetime satisfies the equations of the theory of gravitation (Einstein's vacuum equations).

It is required to find the explicit form of the spacetime metric under all the conditions described.

Let us consider the Shapovalov wave space for the case when there is a coordinate system with respect to which the metric depends on only one null variable

$$dS^2 = g_{ij}\, dx^i dx^j, \qquad g_{ij} = g_{ij}(x^0), \qquad i,j,k = 0...3; \tag{23}$$

where x^0 is a null variable, i.e.

$$dS = 0 \text{ for } dx^0 \neq 0 \text{ and } dx^p = 0, \qquad p,q,r = 1...3.$$

This Shapovalov metric can be represented in the following general form:

$$dS^2 = 2dx^0dx^1 + g_{ab}(x^0)\Big(dx^a + g^a(x^0)dx^1\Big)\Big(dx^b + g^b(x^0)dx^1\Big), \\ a,b,c = 2,3. \tag{24}$$

Substituting the metric (24) in the Einstein's equations $R_{ab} = 0$ it turns out that g^a are constants and the metric can be represented as:

$$dS^2 = 2\,dx^0 dx^1 + g_{ab}(x^0)\,dx^a\,dx^b, \qquad a, b = 2, 3; \tag{25}$$

For the metric (25) there remains only one component of the Ricci tensor R_{00}, that does not vanish identically.

Below, we consider the metric of a plane gravitational wave (25) for the case when spacetime is spatially homogeneous.

Symmetries of plane wave models of spatially homogeneous spacetimes

The model of spacetime considered admits 3 commuting Killing vectors $Y^i_{(p)}$. In a privileged coordinate system we have:

$$Y^i_{(0)} = (0,\, 1,\, 0,\, 0)\,, \tag{26}$$

$$Y^i_{(1)} = (0,\, 0,\, 1,\, 0)\,, \tag{27}$$

$$Y^i_{(2)} = (0,\, 0,\, 0,\, 1)\,, \tag{28}$$

The vector $Y^i_{(0)}$ is a null vector, and the vectors $Y^i_{(1)}$ and $Y^i_{(2)}$ are spacelike vectors.

The one additional Killing vector, which can provides the spatial homogeneity of the model, must belong to two basic types:

$$\text{Type A:} \quad Y^i_{(3)} = \left(-x^0,\, x^1,\, ax^2 + bx^3,\, \tilde{a}x^2 + \tilde{b}x^3\right), \tag{29}$$

$$\text{Type B:} \quad Y^i_{(3)} = \left(1,\, 1,\, ax^2 + bx^3,\, \tilde{a}x^2 + \tilde{b}x^3\right), \tag{30}$$

where a, $\tilde{a}$, b, $\tilde{b}$ are constants.
Then Killing vectors $Y^i_{(1)}$, $Y^i_{(2)}$, $Y^i_{(3)}$ can provide spatial homogeneity of the model.
As follows from Einstein's vacuum equations, type B leads to contradictions. Therefore, below only type **A** is considered.

The additional Killing vector for this class of spacetimes has the following form:

$$Y^i_{(3)} = \left(-x^0,\, x^1,\, \lambda_2 x^2,\, \lambda_3 x^3\right), \tag{31}$$

where x^0 is a null wave variable and λ_2, λ_2 – constants.

The commutation relations for the Killing vectors defining a subgroup of spatial isometry in the class A1 have the form:

$$\left[Y_{(1)}, Y_{(2)}\right] = 0, \tag{32}$$

$$\left[Y_{(1)}, Y_{(3)}\right] = \lambda_2 Y_{(1)}, \tag{33}$$

$$\left[Y_{(2)}, Y_{(3)}\right] = \lambda_3 Y_{(2)}. \tag{34}$$

Spatially homogeneous wave metric class A1

From the Killing equations we obtain the following form of metric

$$
\mathrm{g}_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{{x^0}^{2\lambda_2}}{\sigma^2} & -\frac{\alpha {x^0}^{\lambda_2+\lambda_3}}{\sigma^2} \\ 0 & 0 & -\frac{\alpha {x^0}^{\lambda_2+\lambda_3}}{\sigma^2} & \frac{{x^0}^{2\lambda_3}}{\sigma^2} \end{pmatrix}, \tag{35}
$$

where α, λ_2, λ_3 are constant parameters of model, x^0 is a null wave variable.

$$
g = \det g_{ij} = -\frac{{x^0}^{2(\lambda_2+\lambda_3)}}{\sigma^2}, \qquad \sigma^2 = 1 - \alpha^2,
$$

$$
-1 < \alpha < 1, \qquad 0 < \sigma^2 \leq 1.
$$

Non-flat spacetimes of class A1 are of type VI_a according to Bianchi's classification and type N according to Petrov's classification.

Spatially homogeneous wave models class A2

The additional Killing vector for this class of spacetimes has the following form:

$$Y^i_{(3)} = \left(-x^0,\, x^1,\, \lambda x^2,\, x^2 + \lambda x^3\right), \tag{36}$$

where x^0 is a null wave variable, λ is a constant.
The commutation relations for the class A2 have the form:

$$\left[Y_{(1)}, Y_{(2)}\right] = 0, \tag{37}$$
$$\left[Y_{(1)}, Y_{(3)}\right] = \lambda Y_{(1)} + Y_{(2)}, \tag{38}$$
$$\left[Y_{(2)}, Y_{(3)}\right] = \lambda Y_{(2)}. \tag{39}$$

Spatially homogeneous wave metric class A2

From the Killing equations we obtain the following form of wave metric:

$$
\mathrm{g}_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{{x^0}^{2\lambda}\left(\alpha^2 \ln^2 x^0 - 2\beta \ln x^0 + \gamma^2\right)}{\sigma^2} & \frac{{x^0}^{2\lambda}(\alpha^2 \ln x^0 - \beta)}{\sigma^2} \\ 0 & 0 & \frac{{x^0}^{2\lambda}(\alpha^2 \ln x^0 - \beta)}{\sigma^2} & \frac{\alpha^2 {x^0}^{2\lambda}}{\sigma^2} \end{pmatrix} \tag{40}
$$

where α, β, γ, λ are constant parameters of model, x^0 is a null wave variable.

$$
g = \det g_{ij} = -\frac{{x^0}^{4\lambda}}{\sigma^2}, \qquad \sigma^2 = \alpha^2\gamma^2 - \beta^2,
$$

$$
\alpha\gamma\sigma \neq 0, \qquad 0 \leq \beta^2 < (\alpha\gamma)^2.
$$

The spacetimes of the class A2 are of type IV according to Bianchi's classification and type N according to Petrov's classification.

The additional Killing vector for this class of spacetimes has the following form:

$$Y^i_{(3)} = \left(-x^0,\, x^1,\, \lambda x^2 - x^3,\, x^2 + \lambda x^3\right), \tag{41}$$

where x^0 is a null wave variable, λ is a constant.
The commutation relations for the class A3 have the form:

$$\left[Y_{(1)}, Y_{(2)}\right] = 0, \tag{42}$$

$$\left[Y_{(1)}, Y_{(3)}\right] = \lambda Y_{(1)} + Y_{(2)}, \tag{43}$$

$$\left[Y_{(2)}, Y_{(3)}\right] = -Y_{(1)} + \lambda Y_{(2)}. \tag{44}$$

Spatially homogeneous wave metric class A3

From the Killing equations we obtain the following form of gravitational wave metric g_{ij}:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{{x^0}^{2\lambda}\left(\gamma-\alpha\cos(\ln x^{0^2})-\beta\sin(\ln x^{0^2})\right)}{\sigma^2} & \frac{{x^0}^{2\lambda}\left(\alpha\sin(\ln x^{0^2})-\beta\cos(\ln x^{0^2})\right)}{\sigma^2} \\ 0 & 0 & \frac{{x^0}^{2\lambda}\left(\alpha\sin(\ln x^{0^2})-\beta\cos(\ln x^{0^2})\right)}{\sigma^2} & \frac{{x^0}^{2\lambda}\left(\gamma+\alpha\cos(\ln x^{0^2})+\beta\sin(\ln x^{0^2})\right)}{\sigma^2} \end{pmatrix}$$

where α, β, γ, λ are constant parameters of the model, x^0 is a null wave variable.

$$g = \det g_{ij} = -\frac{(x^0)^{4\lambda}}{\sigma^2}, \qquad \sigma^2 = \gamma^2 - \alpha^2 - \beta^2, \qquad \gamma \neq 0.$$

The spacetimes A3 class are of type VII$_a$ according to Bianchi's classification and type N according to Petrov's classification.

Conclusion on Part III: Shapovalov gravitational wave exact solutions for spatially homogeneous models of Bianchi universes

- A classification of spatially homogeneous plane-wave models of spacetime is constructed.
- Three classes of wave spatially homogeneous exact models of spacetime are obtained (A1, A2, A3).
- The models considered can describe the primordial gravitational waves of the Universe.
- The models considered can be used to obtain exact wave solutions in modified theories of gravity.

The interval s of the spacetime of a gravitational wave can be reduced to the following form in wave coordinates:

$$ds^2 = 2dx^0 dx^1 + \sum_{p,q} g_{pq}(x^0)\Big(dx^p + f^p(x^0)dx^1\Big)\Big(dx^q + f^q(x^0)dx^1\Big), \quad (45)$$

where x^0 is the wave variable, $p, q = 2, 3$.
Einstein's equations of the gravitational field in a vacuum give

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \quad \rightarrow \quad f^p(x^0) = 0. \quad (46)$$

We obtain the metric of a "strong" gravitational wave in a vacuum:

$$ds^2 = 2\,dx^0 dx^1 + \sum_{p,q} g_{pq}(x^0)\,dx^p dx^q. \quad (47)$$

Trajectories of light rays (null geodesic lines). I.

The eikonal equation for the trajectories of propagation of light in a gravitational field with metric $g^{\alpha\beta}$ has the form:

$$\sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial \Psi}{\partial x^{\alpha}} \frac{\partial \Psi}{\partial x^{\beta}} = 0, \tag{48}$$

where Ψ is the eikonal function.
Separating the variables, we obtain

$$\Psi = \psi_0(x^0) + k_i x^i, \tag{49}$$

$$\psi_0(x^0) = -\frac{k_p k_q}{2k_1} G^{pq}(x^0), \qquad G^{pq}(x^0) = \int g^{pq}(x^0)\, dx^0, \tag{50}$$

where the independent constant parameters k_i are determined by the initial values.

Trajectories of light rays (null geodesic lines). II.

Based on the eikonal function, we obtain the trajectories of light rays and massless particles in accordance with the Hamilton-Jacobi formalism in the following form:

$$x^1 = \gamma_1 - \frac{k_p k_q}{2(k_1)^2}\, G^{pq}\big(x^0\big), \tag{51}$$

$$x^p = \gamma_p + \frac{k_q}{k_1}\, G^{pq}\big(x^0\big), \tag{52}$$

$$G^{pq}(x^0) = \int g^{pq}(x^0)\, dx^0.$$

where the constants γ_i are independent parameters of the light trajectories determined by the initial or boundary conditions. The wave variable x^0 plays the role of a parameter in the used privileged coordinate system along the trajectories of massless particles and light rays.

Boundary conditions of light signal propagation

Let us denote the world coordinates of the radiation source as $x^\alpha_{{}_S}$, and the observer coordinates as $x^\alpha_{{}_D}$, then we obtain a system of equations for the trajectory of a light beam connecting the world points $x^\alpha_{{}_S}$ and $x^\alpha_{{}_D}$:

$$x^1_{{}_S} = \gamma_1 - \frac{k_p k_q}{2(k_1)^2}\, G^{pq}\big(x^0_{{}_S}\big), \tag{53}$$

$$x^p_{{}_S} = \gamma_p + \frac{k_q}{k_1}\, G^{pq}\big(x^0_{{}_S}\big), \tag{54}$$

$$x^1_{{}_D} = \gamma_1 - \frac{k_p k_q}{2(k_1)^2}\, G^{pq}\big(x^0_{{}_D}\big), \tag{55}$$

$$x^p_{{}_D} = \gamma_p + \frac{k_q}{k_1}\, G^{pq}\big(x^0_{{}_D}\big), \tag{56}$$

where γ_1, γ_2, γ_3, k_2/k_1 and k_3/k_1 are five independent constant parameters of the light signal trajectory.

From the equations of the trajectory of a light beam connecting world points x^α_S and x^α_D we obtain an expression for the beam parameters k_p/k_1 through the parameters γ_p:

$$\frac{k_p}{k_1} = \sum_q \left[G^{-1}_S\right]_{pq} \left(x^q_S - \gamma_q\right). \tag{57}$$

Then, taking into account (57), we obtain the relation for γ_1 through the parameters γ_p:

$$\gamma_1 = x^1_S + \frac{k_p k_q}{2(k_1)^2}\, G^{pq}\left(x^0_S\right) = x^1_S + \frac{1}{2}\sum_{p,q} \left[G^{-1}_S\right]_{pq} \left(x^p_S - \gamma_p\right)\left(x^q_S - \gamma_q\right) \tag{58}$$

Light signal parameters γ_p

To determine the parameters γ_p we use the relation

$$x_{D}^{p}=\gamma_p+\frac{k_q}{k_1}\,G^{pq}\left(x_{D}^{0}\right)=\gamma_p+\sum_{r,q}G_{D}^{pq}\left[G_{S}^{-1}\right]_{qr}\left(x_{S}^{r}-\gamma_r\right)=$$

$$=\sum_{r}\gamma_r\left(\delta_p^r-\sum_{q}G_{D}^{pq}\left[G_{S}^{-1}\right]_{qr}\right)+\sum_{r,q}G_{D}^{pq}\left[G_{S}^{-1}\right]_{qr}x_{S}^{r} \qquad (59)$$

The obtained relation allows us to determine the parameters γ_p for the light signal through the coordinates of the source and detector, using matrix notation

$$\gamma_p=\sum_{q}\left[\left(I-G_{D}G_{S}^{-1}\right)^{-1}\right]_{pq}\left(x_{D}^{q}-\sum_{r}\left[G_{D}G_{S}^{-1}\right]_{qr}x_{S}^{r}\right), \qquad (60)$$

where $G(x^0)$ is the matrix of integrals of the metric components g^{pq} and I is the identity matrix.

Light signal parameter γ_1

Then for the parameter of the light beam γ_1 we obtain the following expression

$$\gamma_1 = x_S^1 + \frac{1}{2}\sum_{p,q} x_S^p \left[G_S^{-1}\right]_{pq} x_S^q - \frac{1}{2}\sum_{p,q}\left(x_S^p - \sum_r \left[G_D G_S^{-1}\right]_{pr} x_S^r\right) \times$$

$$\times \sum_r \left[(G_S - G_D)^{-1}\right]_{rp} x_S^r$$

$$-\frac{1}{2}\sum_{p,q} x_S^p \left[(G_S - G_D)^{-1}\right]_{pq}\left(x_D^q - \sum_r \left[G_D G_S^{-1}\right]_{qr} x_S^r\right)$$

$$-\frac{1}{2}\sum_{p,q}\left(x_S^p + \sum_r \left[G_D G_S^{-1}\right]_{pr} x_S^r\right)\left[\left(I - G_D G_S^{-1}\right)^{-1}\right]_{pq} \times$$

$$\times\left(\sum_r \left[(G_S - G_D)^{-1}\right]_{pr} x_D^r - \sum_{s,r}\left[(G_S - G_D)^{-1}\right]_{ps}\left[G_D G_S^{-1}\right]_{sr} x_S^r\right) \quad (61)$$

Then the parameters of the light beam k_p/k_1 can be written as:

$$\frac{k_p}{k_1} = \sum_q \left[G_S^{-1}\right]_{pq} x_S^q - \sum_{q,s} \left[G_S^{-1}\right]_{pq} \left[\left(I - G_D G_S^{-1}\right)^{-1}\right]_{qs} \times$$

$$\times \left(x_D^s - \sum_r \left[G_D G_S^{-1}\right]_{sr} x_S^r \right) = \sum_q \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq} x_D^q +$$

$$+ \sum_{q,r} x_S^r \left(\delta_r^q \left[G_S^{-1}\right]_{pq} - \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq} \left[G_D G_S^{-1}\right]_{qr} \right) =$$

$$= \sum_q \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq} x_D^q +$$

$$+ \sum_q \left(\left[G_S^{-1}\right]_{pq} - \left[\left(G_S - G_S G_D^{-1} G_S^{-1}\right)^{-1}\right]_{pq} \right) x_S^q \qquad (62)$$

Finally, the **"delay" equation** of the light signal, relating the coordinates of the source x_S^α and the coordinates of the detector x_D^α in the wave coordinate system ("light cone") will take the form:

$$0 = 2\left(x_D^1 - x_S^1\right) + \sum_{p,q=2}^{3} \left(x_D^p - x_S^p\right) \left[\left(G_D - G_S\right)^{-1}\right]_{pq} \left(x_D^q - x_S^q\right), \quad (63)$$

$$G^{pq}(x^0) = \int g^{pq}(x^0)\, dx^0.$$

In square brackets is the inverse matrix of the difference of matrices $G_D = G^{pq}(x_D^0)$ and $G_S = G^{pq}(x_S^0)$ for the detector and source.

The relation we obtained (63) is for a gravitational wave with the metric $g^{pq}(x^0)$ an analogue of the interval along the trajectory of propagationof light in flat Minkowski spacetime.

Conclusion on Part IV: Light signal delay in a gravitational wave

I. Based on the Hamilton-Jacobi formalism for the spacetime of the exact model of a gravitational wave in a privileged wave coordinate system, an explicit form of **light beams trajectories** in a gravitational wave is found.

II. General relations are obtained that determine the form of the **light "cone"** for the propagation of radiation in a gravitational wave.

III. A general form of relations is found that connect the coordinates of the world points of the radiation source and the coordinates of the radiation detector (i.e., the observer) in a gravitational wave along the trajectory of light connecting these points (**retarded time of radiation equation**).

Summary

I. Shapovalov spacetimes allow constructing exact models of gravitational waves both in Einstein's theory of gravity and in modified theories of gravity.

II. Shapovalov spacetimes allow analytically constructing test particle trajectories and light ray trajectories in gravitational waves.

III. Shapovalov spacetimes allow modeling gravitational waves in Bianchi universes.

IV. A wide range of exact models of gravitational waves with calculation of their observed physical effects (for example, the delay of a light signal in a gravitational wave) is presented.