Finiteness of piecewise flat quantum gravity with matter

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1 Introduction

- QG theory definition:

  Let $M = \Sigma \times [0, t]$, then a QG theory is a map

$$(M, g, \phi) \to (\hat{M}, \hat{g}, \hat{\phi})$$

  such that $M \subseteq \hat{M}$, and there is a well-defined evolution operator $\hat{U}(t)$ given by

$$|\Psi_\Sigma(t)\rangle = \hat{U}(t)|\Psi_\Sigma(0)\rangle\,.$$

- In canonical LQG and ASQG $\hat{M} = M$; in string theory $\hat{M}$ is a loop super-manifold.

- In addition, one should be able to construct the semi-classical states $|\Psi_{\Sigma,p_0,q_0}(t)\rangle$ such that

$$\langle\Psi_{\Sigma,p_0,q_0}(t)|\,\hat{q}\,|\Psi_{\Sigma,p_0,q_0}(t)\rangle = q_0(t)\left[1 + O\left(\frac{\hbar}{S_0(t_0)}\right)\right]\,,$$

  and

$$\langle\Psi_{\Sigma,p_0,q_0}(t)|\,\hat{p}\,|\Psi_{\Sigma,p_0,q_0}(t)\rangle = p_0(t)\left[1 + O\left(\frac{\hbar}{S_0(t_0)}\right)\right]\,,$$

  where

$$S_0(t) = \int_0^t d\tau\,\left(p_0\dot{q}_0 - H(p_0, q_0, \tau)\right)\,,$$

  is the classical action for a classical solution $(p_0(t), q_0(t))$, and t_0 is a timescale of the problem considered.

- Problems of well-known candidate QG theories

  1) $\hat{U}(t)$ not well-defined (non-renormalizability of GR + SM; in ASQG $\hat{U}(t)$ is assumed to exists; in CDT calculations can be done only by a computer and there are no analytical expressions)

  2) semiclassical states not known (problem in LQG)

  3) $q_0(t) \neq q_0^{(GR+SM)}(t)$ (problem of string theory, also in spin-foam approaches)

- PFQG (piecewise flat quantum gravity)

  1) $\hat{M} = T(M)$ and the number of DOF is finite (N edge lengths and matter fields values at n vertices of $T(M)$). Consequently $\hat{U}(t)$ can be defined, since the path integral is a finite-dimensional Riemann integral, which can be made convergent by an appropriate choice of the integration measure [3, 5].

  2) The correct semi-classical limit can be obtained when N is large and edge lengths are small, with an appropriate choice of the PI measure [1, 3]. In this case $T(M) \approx M$ and one can use (GR + SM) QFT with a cutoff $\hbar/L$, where L is the average edge length (fluid dynamics approximation).

2 GR path integral in PFQG

- Let M be a smooth 4-dimensional manifold and let $T(M)$ be a PL (piecewise linear) manifold corresponding to a regular triangulation of M (the dual one simplex is a connected 5-valent graph). Let

$$M = M_1 \sqcup (\Sigma \times I) \sqcup M_2 \,,$$

where

$$\partial M_1 = \partial M_2 = \Sigma \,.$$

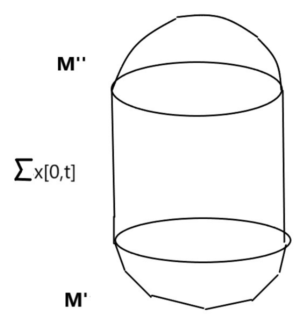

Figure 1: Topology of a PFQG closed spacetime manifold

- When Σ is a non-compact manifold, we maintain a finite DOF by allowing non-zero L_ϵ only for a triangulation of a 3-ball in Σ times an interval $[0,t]$, which is glued to two 4-balls in M_1 and M_2

- Let $\{L_\epsilon | \epsilon \in T_1(M)\}$ be a set of the edge lengths such that $L_\epsilon^2 \in \mathbf{R}$, i.e. $L_\epsilon \in \mathbf{R}_+$ (spacelike edge) or $L_\epsilon \in i\mathbf{R}_+$ (timelike edge).

- A metric on $T(M)$, which is flat in each 4-simplex σ of $T(M)$, is given by

$$G_{\mu\nu}(\sigma) = L_{0\mu}^2 + L_{0\nu}^2 - L_{\mu\nu}^2 \,,$$

where the five vertices of σ are labeled as $0,1,2,3,4$ and $\mu,\nu = 1,2,3,4$ (Cayley-Menger metric).

- The CM metric is not dimensionless and hence it is not diffeomorphic to

$$g_{\mu\nu}(\sigma) = diag(-1,1,1,1) \,.$$

This can be corrected by using a dimensionless PL metric

$$g_{\mu\nu}(\sigma) = \frac{G_{\mu\nu}(\sigma)}{|L_{0\mu}||L_{0\nu}|} \,.$$

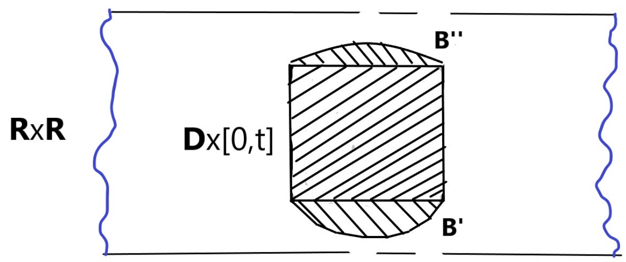

Figure 2: Topology of a PFQG non-compact spatial manifold

- The Einstein-Hilbert (EH) action on M is given by

$$S_{EH} = \int_M \sqrt{|\mathrm{det} g|}\, R(g)\, d^4 x\,,$$

where $R(g)$ is the scalar curvature associated to a metric g. On $T(M)$ the EH action becomes the Regge action

$$S_R(L) = \sum_{\Delta \in T(M)} A_\Delta(L)\, \delta_\Delta(L)\,,$$

when the edge lengths correspond to a Eucledean PL geometry. A_Δ is the area of a triangle Δ, while the deficit angle δ_Δ is given by

$$\delta_\Delta = 2\pi - \sum_{\sigma \supset \Delta} \theta_\Delta^{(\sigma)}\,,$$

where a dihedral angle $\theta_\Delta^{(\sigma)}$ is defined as the angle between the 4-vector normals associated to the two tetrahedrons that share the triangle Δ.

- In the case of a Lorentzian geometry, a dihedral angle can take complex values, so that it is necessary to modify the Regge action formula such that the Regge action takes only the real values.

- This can be seen from the formula

$$\sin\theta_\Delta^{(\sigma)} = \frac{4}{3}\frac{v_\Delta v_\sigma}{v_\tau v_{\tau'}}\,,$$

where $v_s = V_s \geq 0$, if the CM determinant is positive, while $v_s = \mathrm{i}V_s$ if the CM determinant is negative. Consequently, $\sin\theta_\Delta^{(\sigma)} \in \mathbf{R}$ or $\sin\theta_\Delta^{(\sigma)} \in \mathrm{i}\mathbf{R}$. This implies that the Regge action will give a complex number when the spacelike triangles are present.

- One can modify the Regge action as

$$S_R(L) = Re\left(\sum_{\Delta(s)} A_{\Delta(s)}\,\frac{1}{\mathrm{i}}\,\delta_{\Delta(s)}\right) + \sum_{\Delta(t)} A_{\Delta(t)}\,\delta_{\Delta(t)}\,,$$

  where $\Delta(s)$ denotes a spacelike triangle, while $\Delta(t)$ denotes a timelike triangle, so that it is always real and corresponds to the Einstein-Hilbert action on $T(M)$.

- Consequently

$$Z(T(M)) = \int_D \prod_{\epsilon=1}^{N} dL_\epsilon\,\mu(L)\,e^{iS_R(L)/l_P^2}\,,$$

  where $dL_\epsilon = d|L_\epsilon|$ and $\mu(L)$ is a mesure that ensures the finiteness and gives the effective action with a correct semiclassical expansion, see [1, 3]. The integration region D is a subset of $\mathbf{R}_+^N$, consistent with a choice of spacelike and timelike edges.

- $Z(T(M))$ is convergent for the measure

$$\mu(L) = e^{-V_4(M)/L_0^4}\prod_{\epsilon=1}^{N}\left(1+\frac{|L_\epsilon|^2}{l_0^2}\right)^{-p}\,, \qquad (1)$$

  where $p > 1/2$, see [3].

- The bound $p > 1/2$ can be easily derived from the requirement of the absolute convergence

$$|Z| \le \int_D \prod_{\epsilon=1}^{N} dL_\epsilon\,\mu(L) < \prod_{\epsilon=1}^{N}\int_0^\infty dL_\epsilon\,\left(1+\frac{|L_\epsilon|^2}{l_0^2}\right)^{-p}\,.$$

- Note that the convergence can be also obtained without the e^{-V_4/L_0^4} factor in the measure, but the exponential factor is necessary in order to obtain the correct classical limit of the effective action, because when $L_\epsilon \to \infty$, we need

$$\frac{\partial^2 \log\mu(L)}{\partial L_\epsilon \partial L_\epsilon} < 0\,,$$

  see [1, 3, 4].

3 PFQG with the SM matter

- When the SM matter is added, we have

$$S_m = S_H + S_{YM} + S_f + S_Y = \int_M d^4x\,\sqrt{g}\,(\mathcal{L}_H + \mathcal{L}_{YM} + \mathcal{L}_f + \mathcal{L}_Y)\ ,$$

where

$$\mathcal{L}_H = \frac{1}{2}D^\mu\phi^\dagger D_\mu\phi - \lambda_0^2(\phi^\dagger\phi - \phi_0^2)^2\,, \quad \mathcal{L}_{YM} = -\frac{1}{4}Tr\,(F^{\mu\nu}F_{\mu\nu})\ ,$$

$$\mathcal{L}_f = \sum_{k=1}^{48}\epsilon^{abcd}e_b\wedge e_c\wedge e_d\,\bar\psi_k\,(\mathrm{i}\gamma_a(\mathrm{d}+\mathrm{i}\omega+\mathrm{i}g_0A))\,\psi_k\,,$$

$$\mathcal{L}_Y = \sum_{k,l}Y_{kl}\,\langle\bar\psi_k\psi_l\phi\rangle\,, \quad D_\mu\phi = (\partial_\mu + \mathrm{i}\,(g_0A)_\mu)\phi\,,$$

and

$$g_0A = g_{01}A_1 + g_{02}A_2 + g_{03}A_3 \in \text{Lie alg}\,(U(1)\times SU(2)\times SU(3))\ .$$

- On $T(M)$ we have

$$\tilde S_H = \sum_\sigma V_\sigma(L)\,s_{HK} + \sum_\pi V_\pi^*(L)\,s_{HP}\,,$$

where $\pi \in T_0(M)$,

$$s_{HK} = g_\sigma^{\mu\nu}\left(\frac{\phi(\pi_\mu)-\phi(\pi_0)}{|L_{0\mu}|} + \mathrm{i}g_0A_\mu(\pi_0)\phi_{\pi_0}\right)^\dagger\left(\frac{\phi(\pi_\nu)-\phi(\pi_0)}{|L_{0\nu}|} + \mathrm{i}g_0A_\nu(\pi_0)\phi_{\pi_0}\right)$$

and

$$s_{HP} = \lambda_0^2\left(\phi^\dagger(\pi)\phi(\pi) - \phi_0^2\right)^2\ .$$

- The fermion action on $T(M)$ is given by

$$\tilde S_f = \sum_\epsilon V_\epsilon^*(L)\,s_f + \sum_\pi V_\pi^*(L)\ s_{YMf}\,,$$

where

$$s_f = \sum_k \epsilon^{abcd}\,B_{abc}(p)\,\bar\psi_k(\pi)\,\mathrm{i}\gamma_d\,(|L_\epsilon|\,\mathrm{i}\omega_\epsilon(L)\,\psi_k(\pi') + \psi_k(\pi') - \psi_k(\pi))\,,$$

$$s_{YMf} = \sum_k \bar\psi_k(\pi)\,g_0\gamma^\mu(\pi)A_\mu(\pi)\,\psi_k(\pi)\,,$$

and

$$\gamma^\mu(\pi) = e_a^\mu(\pi)\gamma^a\,, \quad e_a^\mu(\pi) = \frac{1}{n_\sigma(\pi)}\sum_{\sigma;\,\pi\in\sigma}e_a^\mu(\sigma)\,.$$

- The Yukawa action on $T(M)$ is given by

$$\tilde S_Y = \sum_\pi V_\pi^*(L)\,s_Y\,,$$

where

$$s_Y = \sum_{k,l}Y_{kl}\langle\bar\psi_k(\pi)\psi_l(\pi)\phi(\pi)\rangle\,.$$

- Therefore the gravity plus matter path integral will be given by

$$Z = \int_D d^N L\, \mu(L)\, e^{iS_R(L)/l_P^2}\, Z_m(L)\,,$$

where

$$Z_m(L) = \int_{D_m} \prod_\alpha d^n\phi_\alpha\, e^{iS_m(\Phi,L)/\hbar}\,,$$

and Φ is a collection of matter fields ϕ_α and n is the number of vertices in $T(M)$.

- Since the convergence of Z_m is not guaranteed, we pass to a Eucledean geometry defined by the edge lengths

$$\tilde{L}_\epsilon = |L_\epsilon|\,,$$

so that all the Eucledean edge lengths are positive real numbers. This is equvalent to a Wick rotation where $\tilde{L}_\epsilon = L_\epsilon$ if ϵ is a spacelike edge and $\tilde{L}_\epsilon = (-\mathrm{i})L_\epsilon$, if ϵ is a timelike edge.

- Then we will consider the integral

$$\tilde{Z}_m(\tilde{L}) = \int_{D_m} \prod_\alpha d^n\phi_\alpha\, e^{-\tilde{S}_m(\Phi,\tilde{L})/\hbar}\,,$$

where $\tilde{S}_m$ is the Euclidian matter action. Since $\tilde{S}_m(\Phi,\tilde{L})$ is a positive function of ϕ, and

$$\tilde{S}_m(\Phi,\tilde{L}) \to +\infty\,, \quad \text{for } |\phi_\alpha| \to +\infty\,,$$

then the integral $\tilde{Z}_m$ will be convergent. Hence we will define

$$Z_m(L) = \tilde{Z}_m(\tilde{L})\Big|_{\tilde{L}=w(L)}\,,$$

where w is the Wick rotation.

- In the case of the SM on $T(M)$ it is useful to write the action as

$$S = S_1 + S_2 + \tilde{S}_2 + S_3\,,$$

where

$$\begin{aligned} S_1 &= r^3\langle\bar{\psi}\psi\rangle + r^4\langle\bar{\psi}\psi A\rangle + r^4\langle\bar{\psi}\psi\phi\rangle \\ S_2 &= r^4\langle(Ar^{-1} + g_0 A^2)^2\rangle \\ \tilde{S}_2 &= r^3\langle\bar{c}(r^{-1} + g_0 Ac)\rangle \\ S_3 &= r^4\langle(\phi r^{-1} + g_0 A\phi)^2\rangle + \lambda_0^2 r^4\langle(\phi^2 - \phi_0^2)^2\rangle\,. \end{aligned}$$

The bracket $\langle XY\cdots\rangle$ represents a sum

$$\sum_{\alpha,\beta,\ldots} c^{\alpha\beta\cdots}(\theta) X_\alpha Y_\beta \cdots\,,$$

and (r,θ) are the spherical coordinates for a vector $\tilde{L}$.

- After integrating the fermions and the ghosts, it can be shown that

$$|Z_m(L)| < r^{c'n} F_n(\theta)\,,$$

where

$$c' = 3c_f - c_b^* = 3c_f - 2|G| - 4 = 260\,,$$

and

$$F_n(\theta) = \int \mathcal{D}\chi\,\mathcal{D}\xi\, e^{-s(\theta,\xi,\chi)} \Delta_{ferm}(\xi,\chi)\Delta_{ghost}(\xi)\,,$$

see [5]. The new variables are given by $\xi = rA$ and $\chi = r\phi$, while $s(\theta,\xi,\chi)$ is the YM action plus the kinetic part of the Higgs action. Δ_{ferm} is the fermionic determinat and Δ_{ghost} is the ghost determinant.

- Consequently

$$|Z| < \int_D d^N L\, \mu(L) |Z_m(L)| < \int d^N L\, \mu(L) r^{c'n} F_n(\theta)\,,$$

so that

$$|Z| < \int_0^\infty r^{N-1+c'n} dr \int_\Omega J_N(\theta)\mu(r,\theta)F_n(\theta) d^{N-1}\theta\,.$$

By using the asymptotic properties of $\mu(r,\theta)$ for small and large r, we obtain

$$|Z| < C_1 \int_0^R r^{c'n+N-1} dr + C_2 \int_R^\infty r^{c'n+N-1-2pN} dr\,.$$

- Hence we can guarantee the absolute convergence of the PFQG path integral if

$$c'n + N(1-2p) < 0\,,$$

so that

$$\frac{c'}{2p-1} < \frac{N}{n}\,.$$

- For a regular triangulation we have

$$\frac{N}{n} \geq \frac{N_1^*}{N_0^*} \geq \frac{5}{2}\,,$$

so that if $c'/(2p-1) < 5/2$, then the absolute convergence bound will be satisfied, which gives

$$p > 52{,}5\,.$$

4 The effective action

- In QFT the EA can be determined from the EA equation

$$e^{i\Gamma[g,\phi]/\hbar} = \int \mathcal{D}h\mathcal{D}\varphi \exp\left(\frac{i}{\hbar}S[g+h,\phi+\varphi] - \frac{i}{\hbar}\int_M \left(\frac{\delta\Gamma}{\delta g(x)}\,h(x) + \frac{\delta\Gamma}{\delta\phi(x)}\,\varphi(x)\right)\sqrt{g}\,d^4x\right).$$

- On $T(M)$ the EA equation becomes

$$e^{i\Gamma(L,\Phi)/\hbar} = \int_{D(L)} d^N l \int_{D_m} d^{cn}\varphi\,\mu(L+l)\,e^{iS(L+l,\Phi+\varphi)/\hbar - i\sum_\epsilon \Gamma'_\epsilon(L,\Phi)l_\epsilon/\hbar - i\sum_\pi \Gamma'_\pi(L,\Phi)\varphi_\pi/\hbar},$$

where c is the number of components of the matter fields ($c = c_f + c_{gh} + c_b = 96 + 24 + 52 = 172$ for the SM) and

$$S(L,\Phi) = \frac{1}{G_N}S_R(L) + S_m(L,\Phi)\,.$$

- The EA equation will be only defined if the gravity plus matter path integral is finite, which is the case for $p > 52{,}5$. This is a consequence of

$$|\tilde{Z}_m(\tilde{L},J)| \leq \tilde{Z}_m(\tilde{L})\,,$$

where

$$\tilde{Z}_m(\tilde{L},J) = \int_{D_m} \prod_\alpha d^n\phi_\alpha\, e^{[-\tilde{S}_m(\Phi,\tilde{L}) + iJ\Phi]/\hbar}\,.$$

- If Γ is not a real solution of the EA equation, then

$$\Gamma \to Re\,\Gamma + Im\,\Gamma\,.$$

4.1 The smooth-manifold approximation

- Let $N\to\infty$ and $|L_\epsilon| = O(1/N)$ in $T(\Sigma\times I)$, such that

$$g_{\mu\nu}(x) \approx g_{\mu\nu}(\sigma)\,, \quad \phi_\alpha(x) \approx \Phi_\alpha(v) \quad \text{for } x\in\sigma \text{ and } v = \text{dual vertex} \in\sigma\,,$$

where $g_{\mu\nu}(x)$ is a smooth metric on $\Sigma\times I$ and $\phi_\alpha(x)$ is a smooth matter field on $\Sigma\times I$. Then

$$\Gamma(L,\Phi) \approx \Gamma_K[g_{\mu\nu}(x),\phi(x)]\,, \quad x\in\Sigma\times I\,,$$

where Γ_K is the QFT effective action for the cutoff $K = 2\pi/\bar{L}$ and $\bar{L}$ is the average edge length in $T(\Sigma\times I)$.

- Consequence of a theorem that a PL function Fourier expansion on an interval $N\bar{L}$ can be approximated by a Fourirer integral with a cutoff $K = 2\pi/\bar{L}$ for N large.

- We also have

$$\frac{\Gamma(L,\Phi)}{\hbar} = \frac{S_R(L) + G_N\,S_m(L,\Phi)}{l_P^2} + \Gamma_1(L,\Phi) + l_P^2\,\Gamma_2(L,\Phi) + l_P^4\,\Gamma_3(L,\Phi) + \cdots\,,$$

$$\approx \frac{\Gamma_K(g,\phi)}{\hbar} = \frac{\frac{1}{G_N}S_{EH}(g) + S_m(g,\phi)}{\hbar} + \Gamma_K^{(1)}(g,\phi) + \hbar\,\Gamma_K^{(2)}(g,\phi) + \hbar^2\,\Gamma_K^{(3)}(g,\phi) + \cdots\,,$$

for $|L_\epsilon| \gg l_P$ and small ϕ, where $\Gamma_K^{(n)}$ is an n-loop QFT effective action for GR coupled to matter, while Γ_n is a coefficient of the perturbative solution in $l_P^2 = G_N\hbar$, see [2].

- Note that $|L_\epsilon| \gg l_P$ still allows for $|L_\epsilon|$ to be microscopically small, so that the smooth-manifold approximation is still valid. For example, the distance probed in the LHC experiments is of the order of 10^{-20}m, while $l_P \approx 10^{-34}$m.

- One can also add the cosmological constant (CC) term to the Regge action, so that

$$S_R(L) \to S_R(L) + \Lambda_c V_4(L)\,.$$

  Then the condition for the semi-classical expansion of the EA, $|L_\epsilon| \gg l_P$ and $L_0 \gg l_P$, is substituted by

$$|L_\epsilon| \gg l_P\,, \quad L_0 \gg \sqrt{l_P L_c}\,,$$

  where $|\Lambda_c| = 1/L_c^2$ [2].

- One can then show that the observed value of the CC belongs to the spectrum of the CC in PFQG, provided that the path integral for gravity + matter is finite, see [2, 4]. Since the PI is finite for $p > 52{,}5$ then the proof given in [4] is now complete.

5 Conclusions

- PFQG defined by the PI measure (1) is the first example of a simple and mathematically complete theory of quantum gravity with the SM matter.

- One can construct the Hartle-Hawking states via the PFQG path integral for the manifold

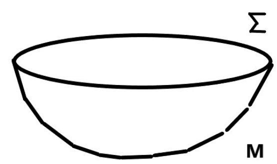

Figure 3: Topology of the Hartle-Hawking manifold

- The Vilenkin states can be constructed from the path integral for the PL manifold $T(\Sigma \times [0,t])$, by taking the limit $L_\epsilon \to 0$ and $\phi_v \to 0$ on the initial surface $T(\Sigma)$.

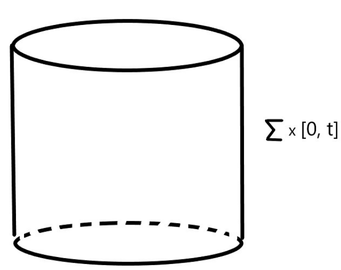

Figure 4: Topology of the QM propagator manifold.

- The EA can be associated to a quantum state via Fig. 5, see [6].

- Physics of PFQG $\approx$ dynamics of the effective action.

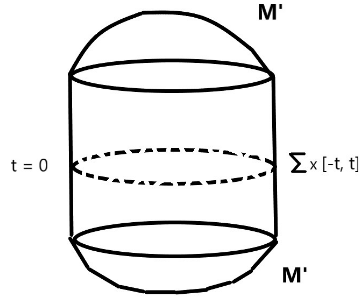

Figure 5: Topology of the effective action manifold

- Perturbative EA $\approx$ long edge-length ($L_\epsilon \gg l_P$) expansion $\approx$ QFT with a cutoff $\hbar K \ll E_P$.
- New physics $\approx$ non-perturbative EA ($L_\epsilon \approx l_P$).
- Non-perturbative EA $\approx$ short edge-length expansion of the EA

$$\Gamma(L,\Phi) \approx \sum_{k_1+k_2+\cdots+k_N \geq 0} l_P^{-(k_1+\cdots+k_N)}\, \gamma_{k_1 k_2 \cdots k_N}(\Phi) L_1^{k_1} L_2^{k_2} \cdots L_N^{k_N} \,.$$

References

[1] A. Miković, *Effective actions for Regge state-sum models of quantum gravity*, Adv. Theor. Math. Phys. 21 (2017) 631

[2] A. Miković and M. Vojinović, *Solution to the cosmological constant problem in a Regge quantum gravity model*, Europhys. Lett. 110 (2015) 40008

[3] A. Miković, *Effective actions for Regge piecewise flat quantum gravity*, Universe 8 (2022) 268

[4] A. Miković and M. Vojinović, *State-sum Models of Piecewise Linear Quantum Gravity*, World Scientific, Singapore (2023)

[5] A. Miković, *Finiteness of quantum gravity with matter on a PL spacetime*, Class. Quant. Grav. 40 (2023) 245011, arXiv:2306.15484

[6] A. Miković, *Physical States and Transition Amplitudes in Piecewise Flat Quantum Gravity*, Int. J. Mod. Phys. D (2024), to appear, arXiv:2405.18893