Courant algebroids in bosonic string theory

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Outline

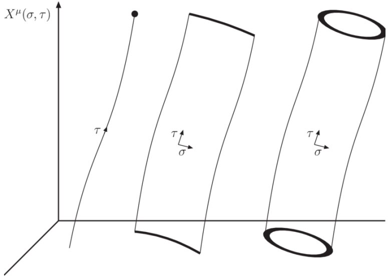
- Introduction to string theory and T-duality
- Introduction to basic notions of generalized geometry
- String symmetries and generalized geometry
- T-dual relations between different twisted Courant brackets

Papers

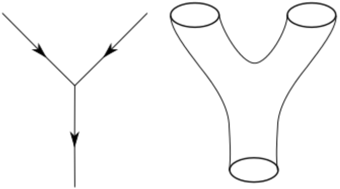
- I Ivanišević, Lj. Davidović, B Sazdović, *Courant bracket found out to be T-dual to Roytenberg bracket, Eur. Phys. J.* **C 80** (2020) 571.
- Lj. Davidović, I. Ivanišević, B. Sazdović, *Courant bracket as T-dual invariant extension of Lie bracket, JHEP* **03** (2021) 109.

String theory

- In string theory, the basic elements of nature are one-dimensional strings, with different particles arising from different vibrations of these strings.
- String theory predicts the existence of a spin-2 boson, which is the **graviton**. For one-loop corrections, gravity obtained this way turns out to be **renormalizable**.



world line, world sheet for open, and world sheet for closed string

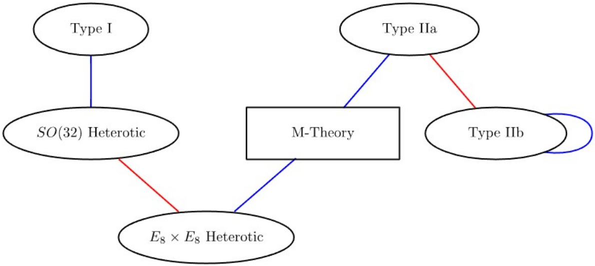


Feynman diagrams for particle vs string interaction

- Bosonic string theory was first developed.

Superstrings and M-theory

- Fermions in string theory are introduced with the help of supersymmetry. At first glance, it appeared that there are five different anomaly-free superstring theories.
- In 1995. Edward Witten suggested that superstring theories are not different, rather that they are connected via web of dualities with one 11-dimensional theory - M-theory.



Dualities: red lines correspond to the T-duality, while blue lines correspond to the S-duality.

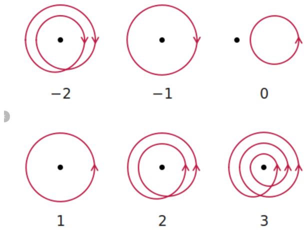
- **T-duality:** duality between theories formulated in different geometries
- **S-duality:** duality betwewen theories with strong and weak coupling constants

T-duality - example: bosonic string in $\mathbb{R}^{1,24} \times S^1$

- Translation along a compactified dimension by a is generated by $e^{ip_{25}a}$. Translation for $a = 2\pi R$ has to correspond to the identity operator, hence we can introduce **momenta numbers** n:

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}.$$

- Algebraic number of times that string winds around the compact dimension is characterized by **winding number** m.



- Mass spectrum

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4}, \quad R \leftrightarrow \frac{l_s^2}{R}, \quad m \leftrightarrow n$$

T-duality

- T-duality is a more general invariance between theories defined in different geometries. It is fundamental part of the string theory, necessary to patch together different superstring theories.
- The first procedure that leads from the initial theory to its T-dual description is the so-called Buscher procedure, which can be applied only in the case of constant background fields.
- In the case of more complex backgrounds, more generalized procedures of T-dualization need to be constructed, and they become increasingly complicated.
- It is believed that to understand T-duality, new mathematical apparatus are necessary, in which different geometries can be easily described. That is why we consider generalized geometry.

Lie algebroid

- **Lie algebroid** consists of a vector bundle E, bracket on the smooth section of E and *anchor* $\rho : E \to T\mathcal{M}$, such that they satisfy the following conditions:

$$\rho[\xi_1, \xi_2] = [\rho(\xi_1), \rho(\xi_2)]_L \,,$$
$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + (\mathcal{L}_{\rho(\xi_1)} f)\ \xi_2 \,,$$
$$[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0 \,.$$

- It is possible to introduce various geometric concepts to different vector bundles.
- Lie derivative:

$$\hat{\mathcal{L}}_\xi f = \mathcal{L}_{\rho(\xi)} f \,, \quad \hat{\mathcal{L}}_{\xi_1} \xi_2 = [\xi_1, \xi_2] \,.$$

- exterior derivative:

$$\begin{aligned}\hat{d}\lambda(\xi_0, ..., \xi_p) &= \sum_{i=0}^{p} (-1)^i \mathcal{L}_{\rho(\xi_i)} \Big(\lambda(\xi_0, ..., \hat{\xi}_i, ..., \xi_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_p) \,,\end{aligned}$$

Example - Koszul bracket

- Anchor: $\theta(\lambda_1)\lambda_2 = \theta(\lambda_1, \lambda_2)\,, \quad \Big(\theta(\lambda_1)\Big)^\mu = \lambda_{1\nu}\theta^{\nu\mu}$
- Koszul bracket
$$[\lambda_1, \lambda_2]_\theta = \mathcal{L}_{\theta(\lambda_1)}\lambda_2 - \mathcal{L}_{\theta(\lambda_2)}\lambda_1 - d(\theta(\lambda_1, \lambda_2))$$
- θ has to be Poisson bi-vector, in order for the structure $\{T^\star\mathcal{M},\ [,]_\theta,\ \theta\}$ to be Lie algebroid
$$\theta([\lambda_1, \lambda_2]_\theta) = [\theta(\lambda_1), \theta(\lambda_2)]_L\,, \quad [\theta, \theta]_S = 0$$
$$[\theta, \theta]_S|^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$$
- Lie derivative:
$$\hat{\mathcal{L}}_\lambda f = \mathcal{L}_{\theta(\lambda)} f\,, \quad \hat{\mathcal{L}}_{\lambda_1}\lambda_2 = [\lambda_1, \lambda_2]_\theta$$
- exterior derivative:
$$(d_\theta f)^\mu = \theta^{\mu\nu}\partial_\nu f\,, \quad (d_\theta\xi)^{\mu\nu} = \theta^{\mu\rho}\partial_\rho\xi^\nu - \theta^{\nu\rho}\partial_\rho\xi^\mu - \xi^\rho\partial_\rho\theta^{\mu\nu}$$

Generalized geometry

- Generalized tangent bundle: $T\mathcal{M} \oplus T^\star\mathcal{M}$
- $O(D,D)$ invariant inner product

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle \xi_1 \oplus \lambda_1, \xi_2 \oplus \lambda_2 \rangle = i_{\xi_1}\lambda_2 + i_{\xi_2}\lambda_1$$

- Let $T\mathcal{M}$ and $T^\star\mathcal{M}$ be vector bundles of two Lie algebroids with brackets $[,]_L$ and $[,]_{L^\star}$. We can define the antisymmetric bracket:

$$\begin{aligned}[\Lambda_1, \Lambda_2] = \quad & \Big([\xi_1, \xi_2]_L + {\mathcal{L}^\star}_{\lambda_1}\xi_2 - {\mathcal{L}^\star}_{\lambda_2}\xi_1 - \frac{1}{2}d^\star(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\ & \oplus\Big([\lambda_1, \lambda_2]_{L^\star} + \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)\end{aligned}$$

- Courant bracket

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}} = [\xi_1, \xi_2]_L \oplus \Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)$$

Courant algebroid

- Courant algebroid $\Big(E, \langle , \rangle, [,], \rho\Big)$. We can define a derivative of a function by $\langle \mathcal{D}f, \Lambda\rangle = \mathcal{L}_{\rho(\Lambda)} f$. The compatibility conditions:

1. $\rho[\Lambda_1, \Lambda_2] = [\rho(\Lambda_1), \rho(\Lambda_2)]_L$
2. $[\Lambda_1, f\Lambda_2] = f[\Lambda_1, \Lambda_2] + (\mathcal{L}_{\rho(\Lambda_1)} f)\Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2\rangle \mathcal{D}f$
3. $\mathcal{L}_{\rho(\Lambda_1)}\langle \Lambda_2, \Lambda_3\rangle = \langle [\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_2\rangle, \Lambda_3\rangle + \langle \Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_3\rangle\rangle$
4. $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$
5. $\mathrm{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = \mathcal{D}\mathrm{Nij}(\Lambda_1, \Lambda_2, \Lambda_3)$

- Jacobiator

$$\mathrm{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) \quad = \quad [[\Lambda_1, \Lambda_2], \Lambda_3] + [[\Lambda_2, \Lambda_3], \Lambda_1] + [[\Lambda_3, \Lambda_1], \Lambda_2]$$

- Nijenhuis operator

$$\mathrm{Nij}(\Lambda_1, \Lambda_2, \Lambda_3) \quad = \quad \frac{1}{6}\Big(\langle [\Lambda_1, \Lambda_2], \Lambda_3\rangle + \langle [\Lambda_2, \Lambda_3], \Lambda_1\rangle + \langle [\Lambda_3, \Lambda_1], \Lambda_2\rangle\Big)$$

Dirac structures

- Dirac structures are subbundles of maximal dimension that are **isotropic** with respect to the inner product and **involutive** with respect to the bracket of the Courant algebroid. A Courant algebroid becomes a Lie algebroid on Dirac structures.
- Isotropy: $\langle \Lambda_1, \Lambda_2 \rangle = 0$
- Dirac structures have a following form:

$$\mathcal{V}_B(\Lambda) = \xi^\mu \oplus 2B_{\mu\nu}\xi^\nu \,, \quad \mathcal{V}_\theta(\Lambda) = \kappa\theta^{\mu\nu}\lambda_\nu \oplus \lambda_\mu \,.$$

- **Example** - Standard Courant algebroid: $\Big(T\mathcal{M} \oplus T^\star\mathcal{M}, \langle , \rangle, [,]_C, \rho = \mathrm{Id}\Big)$
- Pre-symplectic manifolds

$$[\mathcal{V}_B(\Lambda_1), \mathcal{V}_B(\Lambda_2)]_\mathcal{C} = \mathcal{V}_B([\Lambda_1, \Lambda_2]_\mathcal{C}) \,, \quad dB = 0$$

- Poisson manifolds

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_\mathcal{C} = \mathcal{V}_\theta([\Lambda_1, \Lambda_2]_\mathcal{C}) \,, \quad [\theta, \theta]_S = 0$$

Bosonic string σ-model

- Action

$$\mathcal{S} = \kappa \int_\Sigma d\sigma d\tau \Big[B_{\mu\nu}(x) + \frac{1}{2} G_{\mu\nu}(x) \Big] \partial_+ x^\mu \partial_- x^\nu \,, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma$$

- Canonical momentum

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu} \dot{x}^\nu - 2\kappa B_{\mu\nu} x'^\nu \,.$$

- Hamiltonian

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa} \pi_\mu (G^{-1})^{\mu\nu} \pi_\nu + \frac{\kappa}{2} x'^\mu G^E_{\mu\nu} x'^\nu - 2 x'^\mu B_{\mu\rho} (G^{-1})^{\rho\nu} \pi_\nu$$

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$$

- Hamiltonian (matrix form):

$$\mathcal{H}_C = \frac{1}{2\kappa} (X^T)^M H_{MN} X^N$$

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & 2(BG^{-1})_\mu{}^\nu \\ -2(G^{-1}B)^\mu{}_\nu & (G^{-1})^{\mu\nu} \end{pmatrix} \,, \quad X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix} \,.$$

Symmetries of closed bosonic string

- Symmetry generator $\mathcal{H}_{(G,B)} + \{\mathcal{G}, \mathcal{H}_{(G,B)}\} = \mathcal{H}_{(G+\delta G, B+\delta B)}$
- Diffeomorphisms:

$$\mathcal{G}_\xi = \int_0^{2\pi} d\sigma \xi^\mu \pi_\mu$$

$$\begin{aligned} \delta_\xi G_{\mu\nu} &= \mathcal{L}_\xi G_{\mu\nu} = \xi^\rho \partial_\rho G_{\mu\nu} + \partial_\mu \xi^\rho G_{\rho\nu} + \partial_\nu \xi^\rho G_{\rho\mu} \,, \\ \delta_\xi B_{\mu\nu} &= \mathcal{L}_\xi B_{\mu\nu} = \xi^\rho \partial_\rho B_{\mu\nu} - \partial_\mu \xi^\rho B_{\rho\nu} + B_{\mu\rho} \partial_\nu \xi^\rho \,. \end{aligned}$$

- Algebra of diffeomorphisms

$$\left\{ \mathcal{G}_{\xi_1}, \mathcal{G}_{\xi_2} \right\} = -\mathcal{G}_{[\xi_1, \xi_2]_L} \,.$$

- Local gauge transformations:

$$\mathcal{G}_\lambda = \int_0^{2\pi} d\sigma \lambda_\mu \kappa x'^\mu$$

$$\begin{aligned} \delta_\lambda G_{\mu\nu} &= 0 \\ \delta_\lambda B_{\mu\nu} &= (d\lambda)_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu \end{aligned}$$

Symmetries of closed bosonic string

- T-duality relates momenta and winding numbers.

$$P^\mu = \int d\sigma \pi_\mu \,, \quad W^\mu = \int d\sigma \kappa x'^\mu$$

- Two symmetry generators are related by T-duality

$$\kappa x'^\mu \cong \pi_\mu \,.$$

- Motivation to consider a following generator:

$$\mathcal{G}_\Lambda = \mathcal{G}_\xi + \mathcal{G}_\lambda = \int d\sigma \langle \Lambda, X \rangle$$
$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix} \,, \quad X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix}$$

- Generator algebra closes on Courant bracket:

$$\left\{ \mathcal{G}_{\Lambda_1} , \mathcal{G}_{\Lambda_2} \right\} = -\mathcal{G}_{[\Lambda_1, \Lambda_2]_\mathcal{C}}$$

- Courant bracket is an extension of a Lie bracket which is invariant under the T-duality.

Twisted Courant bracket

- The twisted Courant brackets can describe different string fluxes.
- Transformations e^T that keep the inner-product invariant

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle e^T \Lambda_1, e^T \Lambda_2 \rangle \,.$$

- Method for obtaining the twisted Courant bracket from generator algebra:

$$\hat{X}^M = (e^T)^M{}_N \, X^N \,, \quad \hat{\Lambda}^M = (e^T)^M{}_N \, \Lambda^N$$

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X \rangle = \int d\sigma \langle e^T \Lambda, e^T X \rangle = \int d\sigma \langle \hat{\Lambda}, \hat{X} \rangle = \mathcal{G}^{(T)}_{\hat{\Lambda}}$$

- Poisson bracket algebra:

$$\left\{ \mathcal{G}^{(T)}_{\hat{\Lambda}_1}, \mathcal{G}^{(T)}_{\hat{\Lambda}_2} \right\} = -\mathcal{G}^{(T)}_{[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}} \,, \quad [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = e^T [e^{-T} \hat{\Lambda}_1, e^{-T} \hat{\Lambda}_2]_{\mathcal{C}}$$

- Anchor and derivative transformation laws:

$$\rho = \pi \circ e^{-T} \,, \quad \mathcal{D} f = e^T df$$

- All five compatiblity conditions are apriori satisfied.

B-twisted Courant bracket

- B-shifts $e^{\hat{B}}$

$$\hat{B} = \begin{pmatrix} 0 & 0 \\ 2B & 0 \end{pmatrix} , \quad e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix}$$

- We can diagonalize Hamiltonian:

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N = \frac{1}{2\kappa}\hat{X}^M G_{MN} \hat{X}^N$$

$$\hat{X}^M = (e^{\hat{B}})^M{}_N X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix} , \quad G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix}$$

- Non-canonical currents i_μ

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar{\sigma}) , \quad B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

B-twisted Courant bracket

- B-twisted Courant bracket

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = [\xi_1, \xi_2]_L \oplus \Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + dB \Big)$$

- Dirac structures

$$\Big[\mathcal{V}_B(\Lambda_1), \mathcal{V}_B(\Lambda_2)\Big]_{\mathcal{C}_B} = \mathcal{V}_B\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_B}\Big), \quad \forall\, dB$$

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_B} = \mathcal{V}_\theta\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_B}\Big), \quad \mathcal{R} = \frac{1}{2}[\theta, \theta]_S + 2\kappa \wedge^3 \theta\ dB = 0$$

θ-twisted Courant bracket

- T-dual background fields

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}\,, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}$$

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}\,, \quad \theta^{\mu\nu} = \frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$$

- θ-transformations $e^{\hat\theta}$

$$\hat\theta = \begin{pmatrix} 0 & \kappa\theta \\ 0 & 0 \end{pmatrix}\,, \quad e^{\hat\theta} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix}$$

- Hamiltonian in the diagonal form

$$^\star\hat{\mathcal{H}}_C = \frac{1}{2\kappa}\hat X^M\, {}^\star G_{MN}\hat X^N\,, \quad \hat X^M = \begin{pmatrix} \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} k^\mu \\ \pi_\mu \end{pmatrix}$$

$$^\star G_{MN} = \begin{pmatrix} {}^\star(G^{-1})_{\mu\nu} & 0 \\ 0 & {}^\star G^{\mu\nu} \end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G_E^{-1})^{\mu\nu} \end{pmatrix}$$

- Non-canonical currents k^μ

$$\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = -\kappa Q_\rho{}^{\mu\nu}k^\rho\delta(\sigma-\bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma)$$

$$Q_\rho{}^{\mu\nu} = \partial_\rho\theta^{\mu\nu}\,, \qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$$

θ-twisted Courant bracket

- θ-twisted Courant bracket

$$\begin{aligned}
\xi &= [\xi_1, \xi_2]_L - \kappa[\xi_2, \lambda_1\theta]_L + \kappa[\xi_1, \lambda_2\theta]_L + \frac{\kappa^2}{2}[\theta,\theta]_S(\lambda_1, \lambda_2, .) \\
&\quad -\kappa\theta\Big(\mathcal{L}_{\xi_2}\lambda_1 - \mathcal{L}_{\xi_1}\lambda_2 + \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\
\lambda &= \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + \kappa[\lambda_1, \lambda_2]_\theta
\end{aligned}$$

- Dirac structures

$$[\mathcal{V}_B(\Lambda_1), \mathcal{V}_B(\Lambda_2)]_{\mathcal{C}_\theta} = \mathcal{V}_B[\Lambda_1, \Lambda_2]_{\mathcal{C}_\theta}\,, \quad d(BG^{-1}G_E) = 0$$

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_\theta} = 0\,, \quad \forall\theta$$

θ-twisted Courant bracket

- Exchange of the canonical variables and background fields with their respective T-duals:

$$\pi_\mu \leftrightarrow \kappa {x'}^\mu , \quad 2B_{\mu\nu} \leftrightarrow \kappa\theta^{\mu\nu}$$
$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} {x'}^\nu \leftrightarrow \kappa {x'}^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu$$

- Isomorphism between Courant algebroids that correspond to the T-dual transformations

$$\varphi = (e^{\hat\theta} e^{-\hat B})^M{}_N = \begin{pmatrix} \delta^\mu_\nu - 2\kappa(\theta B)^\mu{}_\nu & \kappa\theta^{\mu\nu} \\ -2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} .$$

$$\varphi[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = \Big([\varphi(\Lambda_1), \varphi(\Lambda_2)]_{\mathcal{C}_\theta} \Big) .$$

Conclusions

1. The Courant bracket is a T-dual invariant extension of the Lie bracket. It can be obtained from the Poisson bracket algebra of the generator governing diffeomorphisms and local gauge transformations.
2. We developed a simple method for obtaining arbitrary twisted Courant brackets by changing the basis in which the generator is defined.
3. Twisting the Courant bracket with the Kalb-Ramond field B gives rise to the H-flux, while twisting with the non-commutativity parameter θ yields Q and R fluxes.
4. The Courant bracket twisted by B is T-dual to the Courant bracket twisted by θ.

Thank you for your attention