Modified Gravity and Strong Coupling Issues

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Gravity is very successfully described by the General Relativity theory of Albert Einstein. It is one of the best and most beautiful theories we have. Still, we are stubbornly trying to modify it.

There are mysteries in cosmology. What are the Dark Sectors? Was there inflation, and if yes then how? And if the problems such as H_0 tension are real, what are we making out of that?

On top of that, there are singularities, inherent and unavoidable. They are mostly hidden whenever one can imagine. But don't we want to have a better understanding of what is going on?

And let alone the puzzle of quantum gravity, together with our pathological belief in the mathematically horrendous quantum field theory approach.

And the amazing news we get is that it is extremely difficult to meaningfully modify the theory of General Relativity.

Simple models such as $f(R)$ are almost nothing new, and can be reformulated as an extra universal force mediated by a scalar field on top of the usual gravity. Deeper attempts at modifying it require exquisite care to not encounter with ghosts, or other bad instabilities, or total lack of well-posedness, or no reasonable cosmology available, or.... you name it!

And having the miserable lack of an undoubtful success, it makes all the good sense to try whatever crazy new geometry one can think of. And let it lead us to a better understanding.

On top of the usual curvature, one can consider two other geometric quantities related to the spacetime connection:

$$\text{torsion} \qquad T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$$

and

$$\text{nonmetricity} \qquad Q_{\alpha\mu\nu} = \bigtriangledown_{\alpha} g_{\mu\nu}.$$

Then it is easy to see that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu}\right)$$
$$+ \frac{1}{2}\left(T^{\alpha}{}_{\mu\nu} + T_{\nu}{}^{\alpha}{}_{\mu} + T_{\mu}{}^{\alpha}{}_{\nu}\right) - \frac{1}{2}\left(Q_{\mu\nu}{}^{\alpha} + Q_{\nu\mu}{}^{\alpha} - Q^{\alpha}{}_{\mu\nu}\right).$$

One possible alternative approach is to describe gravity in terms of different geometry.

Metric-compatible teleparallel gravity

In the orthonormal-tetrad-based description of gravity, one can naturally have torsionful connections without curvature or non-metricity by

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{A} \partial_{\mu} e^{A}_{\nu}.$$

Note the zero spin connection! (pure tetrad approach)
At least locally, every connection of this sort can be written like this, for some particular tetrad.

If we go beyond TEGR, or just reproducing GR, this framework is about more than just a metric. In general, different tetrads for the same metric are physically different objects.

Recall that the quest for TEGR action can start from observing that a metric-compatible connection $\Gamma^{\alpha}_{\mu\nu}$ with torsion differs from the Levi-Civita one $\overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}$ by a contortion tensor:

$$\Gamma^{\alpha}_{\mu\nu} = \overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}(g) + K^{\alpha}{}_{\mu\nu}$$

which is defined in terms of the torsion tensor $T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ as

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\alpha\mu\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right).$$

It is antisymmetric in the lateral indices because I ascribe the left lower index of a connection coefficient to the derivative, e.g. $\bigtriangledown_{\mu} T^{\nu} \equiv \partial_{\mu} T^{\nu} + \Gamma^{\nu}_{\mu\alpha} T^{\alpha}$.

The curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}$$

for the two different connections obviously has a quadratic in K expression in the difference. Then making necessary contractions, such as $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$, we can come to

$$0 = R = \overset{(0)}{R} + \mathbb{T} + 2 \overset{(0)}{\bigtriangledown}_\mu T^\mu .$$

Here $T_\mu \equiv T^{\alpha}{}_{\mu\alpha}$ is the torsion vector while the torsion scalar

$$\mathbb{T} \equiv \frac{1}{2} S_{\alpha\mu\nu} T^{\alpha\mu\nu}$$

is given in terms of the superpotential

$$S_{\alpha\mu\nu} \equiv K_{\mu\alpha\nu} + g_{\alpha\mu} T_\nu - g_{\alpha\nu} T_\mu .$$

Due to the basic relation above, the Einstein-Hilbert action $-\int d^4x\sqrt{-g}\overset{(0)}{R}$ is equivalent to the TEGR one, $\int d^4x\|e\|\mathbb{T}$. They are the same, up to the surface term $\mathbb{B} \equiv 2\overset{(0)}{\bigtriangledown}_\mu T^\mu$.

Of course, this equivalence disappears when we go to modified gravity, for example the $f(\mathbb{T})$ gravity:

$$S = \int f(\mathbb{T}) \cdot \|e\| d^4x.$$

Actually, the work of varying this action can be simplified a lot by using this observation.

But many problems await us!
The annoying strong coupling issues...

After some little exercise, the equation of motion can be written as

$$f' \overset{(0)}{G}_{\mu\nu} + \frac{1}{2}\left(f - f'\mathbb{T}\right) g_{\mu\nu} + f'' S_{\mu\nu\alpha}\partial^{\alpha}\mathbb{T} = \kappa \mathcal{T}_{\mu\nu}$$

with $\mathcal{T}_{\mu\nu}$ being the energy-momentum tensor of the matter.
This is a very convenient form of equations!

If $f'' \neq 0$, then the antisymmetric part of the equations takes the form of

$$(S_{\mu\nu\alpha} - S_{\nu\mu\alpha})\partial^{\alpha}\mathbb{T} = 0.$$

It can be thought of as related to Lorentzian degrees of freedom.

And we see that solutions with constant $\mathbb{T}$ are very special and do not go beyond the usual GR, unless we are to study perturbations around them.

The number of degrees of freedom is not very well known. And the main reason is a variable rank of the algebra of Poisson brackets of constraints.
But, what is for sure, is that there must be at least one extra mode.

Still, the trivial Minkowski $e^A_\mu = \delta^A_\mu$ is obviously in a strong coupling regime for a model with $f(0) = 0$ in vacuum. Indeed, then $\mathbb{T} \propto (\partial\delta e)^2$, and for the quadratic action we just take $f(\mathbb{T}) = f_0 + f_1\mathbb{T} + \mathcal{O}(\mathbb{T}^2)$ which means accidental restoration of the full Lorentz symmetry, and linearised GR.

Therefore, the standard properties of gravitational waves are there. This absence of contradiction to experiments is highly problematic.

Cosmology and the degrees of freedom

$f(\mathbb{T})$ gravity is very popular for cosmology with a simple solution of the form

$$ds^2 = a^2(\tau)\left(d\tau^2 - dx^i dx^i\right)$$

in terms of the following tetrad Ansatz:

$$e^A_\mu = a(\tau) \cdot \delta^A_\mu.$$

How can one do cosmological perturbations?

It is not enough to choose just some possible tetrad for the most general perturbed metric like

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= 0\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta + v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a + \partial_{aj}^2\sigma + \partial_j c_a + \frac{1}{2}h_{aj}\right).
\end{aligned}
$$

Instead, one must use the most general Ansatz for the tetrad perturbation

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= a(\tau)\cdot(\partial_i\beta + u_i)\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta + v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a + \partial_{aj}^2\sigma + \epsilon_{ajk}\partial_k s + \partial_j c_a + \epsilon_{ajk}w_k + \frac{1}{2}h_{aj}\right).
\end{aligned}
$$

Under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$ with ξ^0 and $\xi^i \equiv \partial_i \xi + \tilde{\xi}_i$, one can simply derive the following transformation laws:

$$\begin{array}{rcl}
\phi & \longrightarrow & \phi - \xi^{0\prime} - H\xi^0 \\
\psi & \longrightarrow & \psi + H\xi^0 \\
\sigma & \longrightarrow & \sigma - \xi \\
\beta & \longrightarrow & \beta - \xi^0 \\
\zeta & \longrightarrow & \zeta - \xi' \\
c_i & \longrightarrow & c_i - \tilde{\xi}_i \\
v_i & \longrightarrow & v_i - \tilde{\xi}_i'.
\end{array}$$

Gauge invariant combinations and possible gauge choices are obvious.

After careful calculations, there are no new dynamical modes in linear perturbations!
Out of 6 new variables: 5 constrained variables and 1 dropping off, from every equation ("remnant symmetry"?).

Therefore, no new degrees of freedom at the linear level. Hence, the strong coupling problem. Predictions are not reliable.

A very interesting unreliable prediction is non-zero gravitational slip:

$$\phi - \psi = -\frac{12 f_{TT} H (H' - H^2)}{f_T} \zeta$$

where

$$\triangle \zeta = -3 \left(\psi' + H\phi - \frac{H' - H^2}{H} \psi \right).$$

In order to see the extra dynamical modes, one might go for other backgrounds.

Due to the "remnant symmetry", we can take another solution for Minkowski metric

$$e^A_\mu = \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) & 0 & 0 \\ \sinh(\lambda) & \cosh(\lambda) & 0 & 0 \\ 0 & 0 & \cos(\psi) & -\sin(\psi) \\ 0 & 0 & \sin(\psi) & \cos(\psi) \end{pmatrix}$$

with arbitrary functions $\lambda(t,x,y,z)$ and $\psi(t,x,y,z)$.

It has $\mathbb{T} = 0$ and is a solution as long as $f(0) = 0$.

For linear Lorentzian perturbations one gets equations for the perturbations of $\mathbb{T}$

$$\begin{aligned}
-\psi_z \mathbb{T}_t - \lambda_y \mathbb{T}_x + \lambda_x \mathbb{T}_y + \psi_t \mathbb{T}_z &= 0,\\
\psi_y \mathbb{T}_t - \lambda_z \mathbb{T}_x - \psi_t \mathbb{T}_y + \lambda_x \mathbb{T}_z &= 0,\\
-\lambda_y \mathbb{T}_t - \psi_z \mathbb{T}_x + \lambda_t \mathbb{T}_y + \psi_x \mathbb{T}_z &= 0,\\
-\lambda_z \mathbb{T}_t + \psi_y \mathbb{T}_x - \psi_x \mathbb{T}_y + \lambda_t \mathbb{T}_z &= 0.
\end{aligned}$$

In generic enough a situation we get $\mathbb{T} =$const. However, in case of only a boost or only a rotation, perturbations of non-constant $\mathbb{T}$ are possible.

In particular, for $\lambda(z)$ and no rotation, we get a new mode with strange Cauchy data of $C_1(y,z)$ and $C_2(x,y,z)$.

The Hamiltonian analysis of $f(\mathbb{T})$ is tricky. There are two contradictory claims (I give it in 4D):
1. It has 5 d.o.f., i.e. three extra propagating degrees of freedom. (Li, Miao, Miao 2011; Blagojevic, Nester 2020)
2. It has 3 d.o.f., i.e. one extra propagating degree of freedom. (Ferraro, Guzman 2018)

The last version of the first claim is probably the most accurate one, even though not without its shortcomings. In particular, no attention is payed to singular surfaces in the phase space, jumps in the ranks of Poisson brackets algebra, and so on.
At the same time, to the best of my knowledge, no one has ever seen the full set of three new modes in practical calculations.

The main mistake in the second claim was in neglecting the spatial derivatives of $\mathbb{T}$ in the Poisson brackets.
And indeed, our "almost one" new mode was seen around the non-trivial Minkowski background with $\mathbb{T} = 0$.

In cosmological tasks, the $\mathbb{T}$ scalar does naturally have a time-like gradient, and therefore can be taken for a time variable.
Does this mean a possible existence of preferred foliation in this case?

Some more discussion on these teleparallel issues see in my conference (XII Bolyai-Gauss-Lobachevsky Conference (BGL-2024): Non-Euclidean Geometry in Modern Physics and Mathematics) proceedings paper
A. Golovnev. *Degrees of Freedom in modified Teleparallel Gravity*. Ukrainian Journal of Physics **69** (2024) 456

Riemannian-geometry-based modified gravity

It's probably enough of complicated stuff. Let's look at more elementary models, purely in terms of Riemannian geometry.

One of the nicest modifications is $f(R)$ gravity

$$S[g_{\mu\nu}] = \int \sqrt{-g} \cdot f(R).$$

If to Taylor-expand around zero curvature, one can talk about
pure quadratic $f(R) = R^2$
and full quadratic $f(R) = R + R^2$ cases.

Recently, in the paper [A. Hell, D. Lust, G. Zoupanos. *On the Degrees of Freedom of R^2 Gravity in Flat Spacetime*. Journal of High Energy Physics JHEP02(2024)039] the gravity model of "pure R^2" type was considered. Namely, the Lagrangian density is R^2 with no linear term. Two different results were reported about the linearised around Minkowski space limit.

Substituting the standard parametrisation of cosmological perturbation theory into the action, no dynamical modes survive the limit.

With the spin-projector parametrisation of

$$\begin{aligned} h_{\mu\nu} = & \quad \text{vector and tensor contributions} \\ & + \left(\partial_\mu \partial_\nu - \frac{1}{4}\eta_{\mu\nu}\Box\right)\mu + \frac{1}{4}\eta_{\mu\nu}\lambda \end{aligned}$$

a scalar mode seems to survive...

A simple illustration

The point is that making a derivative substitution or fixing a gauge directly inside the action does generally change the model at hand!

As a very simple example, let's look at the Proca field:
$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$.

If we parametrise $A_\mu = A_\mu^T + \partial_\mu \phi$ with $\partial^\mu A_\mu^T \equiv 0$, it is all right at the level of equations of motion. However, being substituted into the action, it changes the model due to higher derivative nature of the longitudinal mode.

If we do it as $A_\mu = \left(A_0 \ , \ A_i^{\mathcal{T}} + \partial_i \phi\right)$ with $\partial_i A_i^{\mathcal{T}} \equiv 0$, similar to cosmological perturbations, then it is fully all right if in perturbation theory due to conditions at spatial infinity.

The idea of the Stückelberg trick is that instead of one vector field we may work with one vector and one scalar, $A_\mu \longrightarrow A_\mu + \partial_\mu \phi$, such that the new system enjoys a gauge symmetry of
$\phi \longrightarrow \phi + \chi, \qquad A_\mu \longrightarrow A_\mu - \partial_\mu \chi.$
The action then takes the form of

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} \left(A_\mu A^\mu + 2A^\mu \partial_\mu \phi + (\partial^\mu \phi) \partial_\mu \phi \right)$$

with the equations of motion

$$\partial_\nu F^{\nu\mu} + m^2 \left(A^\mu + \partial^\mu \phi \right) = 0, \qquad \Box \phi + \partial_\mu A^\mu = 0.$$

All in all, the physical vector $A_\mu + \partial_\mu \phi$ has got absolutely the same dynamics as the vector of the initial non-covariant model.

Suppose that, instead of introducing a new variable, we do a reparametrisation of $A_\mu = A^T_\mu + \partial_\mu \phi, \qquad \partial^\mu A^T_\mu \equiv 0$. This reparametrisation is a bit redundant. In case of separating a 3D-longitudinal mode, we may fix such redundancy by spatial boundary conditions, but we always treat the time differently. It can also be thought of as an (incomplete) gauge fixing in the model defined by the Stückelberg trick. This is all right at the level of equations of motion.

However, being done directly inside the action, it leads to

$$\delta S = \int d^4x \left(\left(\partial_\nu F^{\nu\mu} + m^2 A^{T\mu} \right) \delta A^T_\mu - m^2 (\Box\phi) \delta\phi \right).$$

It describes more freedom than before, even for the physical variable $A^T_\mu + \partial_\mu \phi$.

Coming back to pure R^2 gravity around Minkowski metric...

For $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, to the linear order, the equation

$$2R \cdot R_{\mu\nu} - \frac{1}{2} R^2 \cdot g_{\mu\nu} + 2 \left(g_{\mu\nu} \Box - \bigtriangledown_\mu \bigtriangledown_\nu \right) R = 0$$

reduces to $\partial_\mu \partial_\nu R = 0$ for the scalar $R = \left(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box \right) h^{\mu\nu}$.

We then see that the only information is that $\triangle h_{00}$ is uniquely given in terms of other metric components, up to a possible global addition of $b + c_\mu x^\mu$. Up to the freedom of harmonic functions too, it fixes one of the variables.
Therefore, indeed, there are no dynamical degrees of freedom, one constrained physical mode, and all the rest is pure gauge.
In this limit...

In the language of cosmological perturbation, the vector and tensor ones are obviously in the pure gauge sector, and only the scalars survive. It can be taken as

$$g_{\mu\nu}dx^\mu dx^\nu = (1+2\phi)dt^2 - (1-2\psi)\delta_{ij}dx^i dx^j$$

with $R = 6\Box\psi + 2\triangle(\phi+\psi)$.

In philosophy of perturtbation theory, taking the $\partial_\mu\partial_\nu R = 0$ equation as simply $R = 0$, we get

$$3\Box\psi + \triangle(\phi+\psi) = 0.$$

With no more information available, we should take ψ as also a pure gauge, and ϕ then being the only physical mode, which is constrained.

Nota an interpretational aspect of it, though!

A spin-projector type of the trick can be taken as

$$h_{\mu\nu} = \quad \text{vector and tensor contributions} \quad + \partial_\mu \partial_\nu \Sigma + \eta_{\mu\nu} \Psi.$$

We can immediately calculate $R = -3\Box\Psi$.
Even if, in perturbation theory, we treat the equation as $R = 0$, we suddenly get a dynamical mode of $\Box\Psi = 0$.

The reason is that this representation of the metric is very ambiguous. Let's, for example, take the cosmological perturbations metric above (which did not lead to any dynamical mode) and find the trace and the double divergence of its perturbation for presenting it in the spin-projector shape. We get
$\Box\Sigma + 4\Psi = 2\phi - 6\psi \qquad \text{and} \qquad \Box^2\Sigma + \Box\Psi = 2\ddot{\phi} + 2 \triangle \psi.$
In other words, the two Cauchy data for Ψ simply correspond to a freedom of choosing presentation of the metric in this shape.

At the same time, the action of $\int R^2$ turns into $9\int d^4x \cdot (\Box\Psi)^2$ and produces even a 4-th order equation of motion $\Box^2\Psi = 0$ which cannot be explained by ambiguity of parametrisation.
In this case, the field Ψ was not given any extra derivatives in the action, however the operator of $\partial_\mu\partial_\nu$ had effectively been removed from it producing $\Box R = 0$ instead of $\partial_\mu\partial_\nu R = 0$. Therefore, we get more solutions than the initial model used to have. It is yet another example of problems with substitutions right into the action.

More discussion and various examples are in my paper
A. Golovnev. *On the degrees of freedom count on singular phase space submanifolds*. International Journal of Theoretical Physics **63** (2024) 212

A few words against extended Hamiltonians

For electrodynamics, the total Hamiltonian density

$$\mathcal{H}_T = \frac{1}{2}\pi_i^2 + \pi_i \partial_i A_0 + \frac{1}{4}F_{ij}^2 + \lambda\pi_0,$$

with λ being a Lagrange multiplier, produces then the equations of motion

$$\left\{ \begin{array}{c} \dot{A}_0 = \lambda \\ \dot{A}_i = \pi_i + \partial_i A_0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{c} \dot{\pi}_0 = \partial_i \pi_i \\ \dot{\pi}_i = \partial_j F_{ji} \end{array} \right. \quad \begin{array}{c} \text{with the constraint} \\ \pi_0 = 0. \end{array}$$

As long as we take the primary constraint as an equation rather than an initial datum, this system totally reproduces the Lagrangian dynamics of A_μ as well as the definition of momenta $\pi_0 = 0$ and $\pi_i = \dot{A}_i - \partial_i A_0 = F_{0i}$.

The extended Hamiltonian density is

$$\mathcal{H}_E = \frac{1}{2}\pi_i^2 + \pi_i \partial_i A_0 + \frac{1}{4}F_{ij}^2 + \lambda\pi_0 + \tilde{\lambda}\partial_i\pi_i$$

with the equations

$$\left\{ \begin{array}{c} \dot{A}_0 = \lambda \\ \dot{A}_i = \pi_i + \partial_i A_0 + \partial_i \tilde{\lambda} \end{array} \right. \qquad \left\{ \begin{array}{c} \dot{\pi}_0 = \partial_i \pi_i \\ \dot{\pi}_i = \partial_j F_{ji} \end{array} \right. \qquad \left\{ \begin{array}{c} \pi_0 = 0 \\ \partial_i \pi_i = 0. \end{array} \right.$$

We've got twice the correct amount of gauge freedom, and totally got rid of the Gauß law. The definition of the spatial components' momenta is also lost.

What can be done is a total redefinition of the model. We say that F_{ij} are still considered physical, but not the mixed components of F_{0i}. Instead of the latter we take π_i as physical, forgetting its definition in terms of the fields. If in the Lagrangian equations we substitute F_{0i} by π_i, it yields $\partial_i \pi_i = 0$ (Gauß law in absence of charges) and $\dot{\pi}_i = \partial_j F_{ji}$.

The same happens in general. We artificially enhance the space of variables by pronouncing the momenta independent. After that we add yet another gauge symmetry by simply calling anything unphysical as long as it does not commute with any of the first-class constraints. If the constraints are of the first class, then in each canonical pair of variables they can involve only one combination and transform another one. Therefore, there must exist a gauge-invariant combination which can then serve as a substitute for the corresponding field. And by the very definition of the new gauge symmetry, the dynamics of invariant variables are not changed by adding the secondary constraints to the Hamiltonian.

The same is about GR. There are six physical modes, two of which are dynamical and four constrained. One can say that the four constrained modes get rewritten as combinations with momenta, but then it contradicts the geometric picture of test particles following the geodesics.

When generalising a model or studying problematic loci of the phase space, one should not forget about the differences. If there was an accidental gauge symmetry in a linearised model, extending the Hamiltonian would take us even farther away from the full model. This is not surprising, of course, because we then artificially extend the accidental symmetry which is not present in the full theory.

Conclusions

Strong coupling cases are problematic but very interesting at the same time.

We need a better understanding of what is going on, especially for any progress in modified teleparallel gravity research.

One of the most important tasks is to properly understand the Hamiltonian mechanics of constrained systems, not just as a mere recipe.

Thank you!