Frobenious structures and restricred (2+1)-TQFTs

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- Studying QFT is hard: pertubative methods, renormalization, infinite numer of degrees of freedom. Usual guide: symmetries!

- There are cases in condensed matter physics when certain properties of a gapped systems depend only on topology!

- We can ask for a nontrivial field theory that still works with finite dimensional Hilbert spaces.

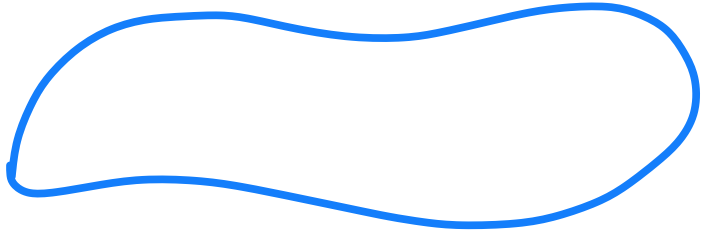
$\Rightarrow$ TQFT

- One physical example: Chern Simons! Take example of SU(2) gauge group and A is a gauge connection, that is locally a Lie algebra valued one form.

$$S = \int \mathrm{Tr}\left(A \wedge \mathrm{d}A + A \wedge A \wedge A\right) \qquad \Rightarrow F = 0$$

- Is this theory trivial? No! It is used in condensed matter do describe QHE, in knot theory to compute Jones polynomials and in 3D gravity.

- Wilson loop observable → can detect nontrivial topology (anyons)!



$$\mathrm{Tr}_R \underbrace{P e^{\oint A}}_{\text{holonomy}}$$

- Previous consideration was classical, can we go to quantum level?
- We will use a formal mathematical approach, first introduced by Atiyah [Atiyah '89].
- A TQFT is a (strong) symmetrical monoidal funtor from the category of cobordisms to the category of vector spaces.
- This approach has an advantage that is well accessible to pure mathematics; it does not require one to deal with path integrals.
- But, what does this means?

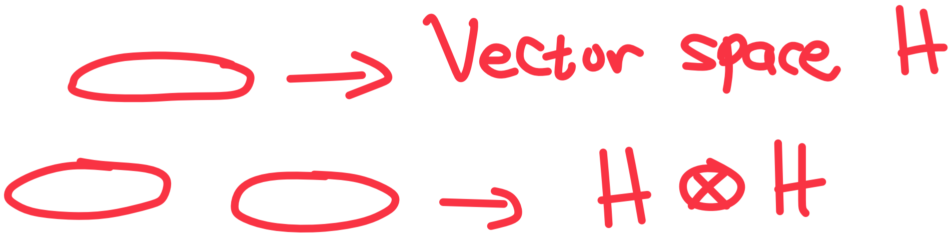
$$\int DA_\mu \, e^{-S[A_\mu]}$$

Def: A category consist of objects $(a, b, c, \ldots)$ and arrows $(f, g, h, \ldots)$, together with rules:

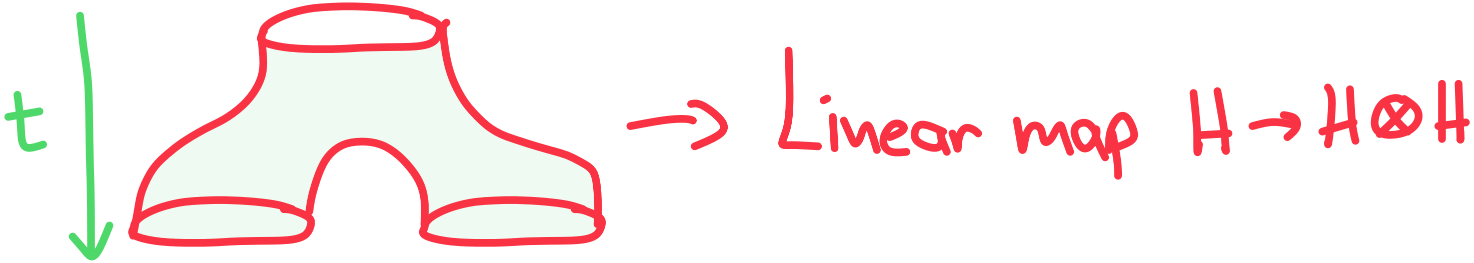
- For every arrow f, there exist objects $\mathrm{dom}(f)$, $\mathrm{cod}(f)$
- For arrows $f : a \to b$ and $g : b \to c$ there is an arrow $g \circ f : a \to c$
- For every object a there is an arrow $\mathbf{1_a} : a \to a$
- Composition is associative
- For every arrow $f : a \to b$ we have $f \circ \mathbf{1_a} = \mathbf{1_b} \circ f$

- Important example: $\mathbf{Vect}_{\mathbb{C}}$: objects are complex vector spaces, arrows are linear maps between vector spaces.
- Functor: mapping between categories (think of group representations).

- Let us illustrate everything in 2D:

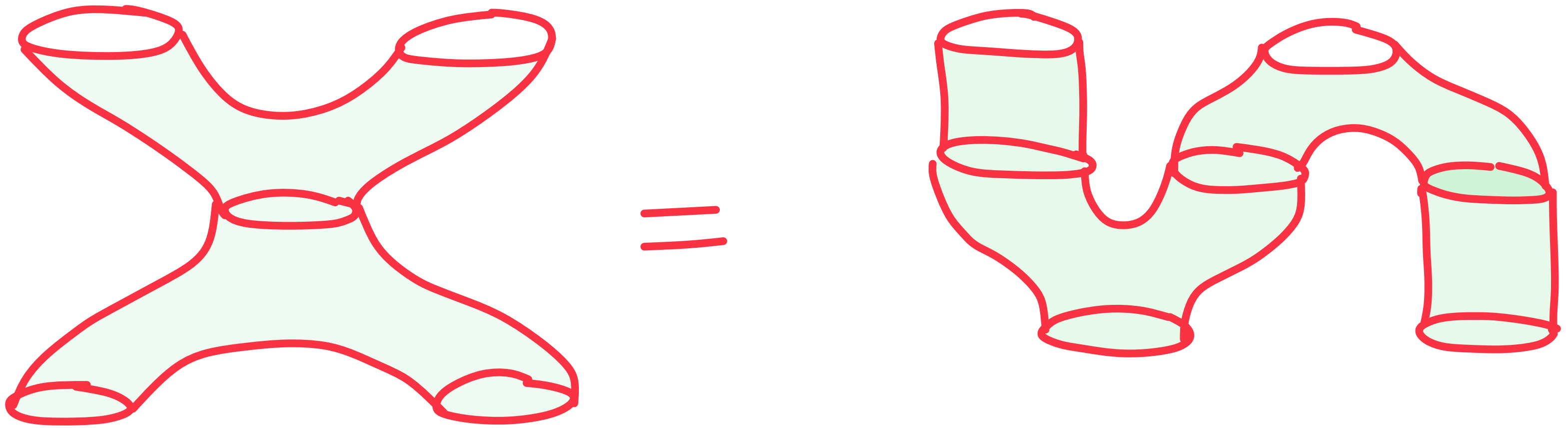


- Cobordisms!



- In two dimensions, it is well-known that we can generate the whole theory from the following cobordisms

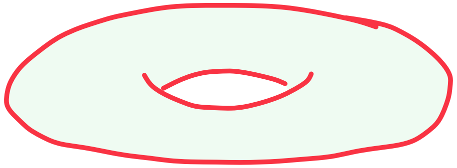
- Everything should depend only on “topology”!



→ Commutative Frobenius algebra

- What happens in three dimensions?

- Three manifolds are much harder to study. Also, objects of **3Cob** are closed, oriented surfaces, with much richer structure than S^1.



- The main character in our story will be torus, so lets focus on it. First, define a mapping class group as the group of orientation preserving homeomorphisms modulo isotopy.

- It is a well known fact that, for torus, this group is $SL(2,\mathbb{Z})$.

- Righthanded Dehn twists on a:

- Matrix representation

$$D_a := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D_b := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

- Another useful set of generators of $SL(2,\mathbb{Z})$, satisfying $S^2 = (ST)^3, S^4 = \mathrm{Id}$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- Motivation of our work: faithful TQFT. In dimension one, almost all TQFTs are faithful [Telebaković Onić '20]. In two dimensions, there exists a faithful TQFT [Telebaković Onić, Petrić, Gajović '20]. What happens in three dimensions?

- Part of this story is well-known: extended TQFT. From physical perspective, they are interesting and important. Mathematically, they are based on a structure of a modular tensor category. Example: Reshetikin-Turaev (RT) TQFT.

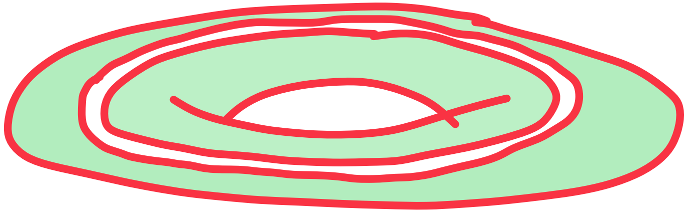
- Unfortunately, those TQFTs are not faithful [Funar '12], there exist some torus bundles thar are not distinguished by RT invariants. We therefore seek to find some representations of $SL(2,\mathbb{Z})$ that are not connected with MTC.

- Actually, similar to the clasification of 2D TQFTs in terms of Frobenious algebras, 3D TQFTs can be clasified using J algebras [Juhasz '18]. However, J algebras are much harder to work with and we therefore seek to define a restricted TQFT.

- We consider only a subcategory of **3Cob**. Today, we focus on βMCG.

$$\beta \circ (D_a \otimes \mathbf{1}) = \beta \circ (\mathbf{1} \otimes D_b) \,.$$
$$\beta \circ (D_b \otimes \mathbf{1}) = \beta \circ (\mathbf{1} \otimes D_a) \,.$$

- General hint: thick torus!



- Here we have: the identity, the pairing β, the copairing γ and the mapping cylinder.

- Examples of representations of $SL(2,\mathbb{Z})$ that are able to distinguish between torus bundles:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or •••

Construction of restricted TQFT

- Examples of commutative Frobenius algebras: centres of group algebra!

  For D_8 group, we get the following multiplication.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

- We take the following representation for MCG:

$$D_a = \begin{pmatrix} 0 & 0 & -\frac{e^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 & -\frac{1}{2}ie^{-\frac{i\pi}{6}} & 0 & 0 \\ \frac{e^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{i\pi}{6}} & 0 & -i\sqrt{2}e^{-\frac{i\pi}{6}} & -\frac{e^{-\frac{i\pi}{6}}}{\sqrt{2}} \\ 0 & -\frac{1}{2}e^{-\frac{i\pi}{6}} & e^{-\frac{i\pi}{6}} & 0 & -\frac{ie^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{2}e^{-\frac{i\pi}{6}} & 0 & -2e^{-\frac{i\pi}{6}} & -ie^{-\frac{i\pi}{6}} \\ 0 & -\frac{ie^{-\frac{i\pi}{6}}}{2\sqrt{2}} & -\frac{ie^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 & -\frac{1}{2}e^{-\frac{i\pi}{6}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{e^{-\frac{i\pi}{6}}}{2\sqrt{2}} & 0 & -\frac{1}{2}ie^{-\frac{i\pi}{6}} & 0 \end{pmatrix}$$

→ Works!

$$D_b = \begin{pmatrix} 0 & \sqrt{2}e^{-\frac{i\pi}{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}e^{-\frac{i\pi}{6}} & 0 & -\frac{ie^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 \\ -\frac{e^{-\frac{i\pi}{6}}}{2\sqrt{2}} & 0 & 0 & e^{-\frac{i\pi}{6}} & 0 & -i\sqrt{2}e^{-\frac{i\pi}{6}} & 0 \\ 0 & 0 & e^{-\frac{i\pi}{6}} & 0 & -\frac{ie^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 & -\frac{e^{-\frac{i\pi}{6}}}{\sqrt{2}} \\ -\frac{1}{2}ie^{-\frac{i\pi}{6}} & 0 & 0 & -i\sqrt{2}e^{-\frac{i\pi}{6}} & 0 & -2e^{-\frac{i\pi}{6}} & 0 \\ 0 & 0 & -\frac{ie^{-\frac{i\pi}{6}}}{\sqrt{2}} & 0 & -\frac{1}{2}e^{-\frac{i\pi}{6}} & 0 & -\frac{1}{2}ie^{-\frac{i\pi}{6}} \\ 0 & 0 & -\frac{e^{-\frac{i\pi}{6}}}{2\sqrt{2}} & 0 & -\frac{1}{4}ie^{-\frac{i\pi}{6}} & 0 & 0 \end{pmatrix}$$

- We indeed have a restricted TQFT. Moreover, consider two torus bundles, determined by two matrices

$$A = \begin{pmatrix} 1 & 21 \\ 21 & 442 \end{pmatrix}, \qquad B = \begin{pmatrix} 106 & 189 \\ 189 & 337 \end{pmatrix}$$

- We can easily calculate that we have

$$1 = \beta \circ (\rho(A) \otimes \mathbf{1}) \circ \gamma \neq \beta \circ (\rho(B) \otimes \mathbf{1}) \circ \gamma = 2$$

- We can distinguish between two torus bundles that are not distinguished by RT invariants.

- Along these lines, we wish to introduce the unit ε and the counit η (resulting in $\beta\epsilon\mathbf{MCG}$ category, where lens spaces play a role), and finally multiplications and comultiplications (resulting in a $\mu\epsilon\mathbf{MCG}$ category).

- In the former case, one can show that the aditional relations are

$$\beta \circ (\mathbf{1} \otimes \varepsilon) = \eta$$
$$D_a \circ \varepsilon = \varepsilon$$

- The way we construct those restricted TQFT's is similar to a dimensioanl reduction from three to two dimensions, but with one important difference. We do not assume that our spacetime is of the form $\Sigma \times S^1$, but rather we have a nontrivial a la Hopf fibration.

Thank you for your attention!