Deformed Heisenberg Algebras in Quantum Cosmology

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Outline

Montani, **GB** et al., *Phys. Rev. D* **99**, 063534 (2019).
Giovannetti, **GB**, Mandini, Montani, *Universe* **8** (6), 302 (2022).
GB, Giovannetti, Montani, *IJGMMP* **19** (07), 2250097 (2022).
GB, Montani, Melchiorri, *Phys. Rev. D* **108**, 063505 (2023).

We use natural units $\hbar = c = 8\pi G = 1$.

Classical Hamiltonian Cosmology

The cosmological action and Hamiltonian contain two parts:

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{2} + \mathcal{L}_m \right)$$

$$\mathcal{H} = \mathcal{H}_G(a, p_a) + \mathcal{H}_M(\rho, \phi, ...) = 0$$

From EoM, find Friedmann equation linking expansion to matter:

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3}} \qquad P = w\rho \qquad \rho = \rho(a)$$

Solve Friedmann eq. for scale factor $a(t)$ (we often use $v = a^3$):

$$a(t) \propto (t^2)^{\frac{1}{3(1+w)}} \qquad v(t) \propto (t^2)^{\frac{1}{1+w}}$$

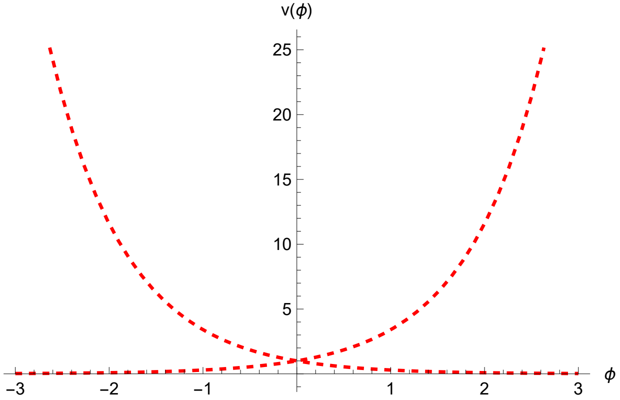
Big Bang/Big Crunch singularities at $t = 0$.

Time and the Scalar Field

$\mathcal{H} = 0 \qquad \Rightarrow \qquad$ Freedom of choosing a time variable.

A good candidate is a free scalar field ϕ (monotonic with t).

Singularities moved at $\phi = \pm\infty$: $\quad v(\phi) \propto \exp\left(\pm\sqrt{\frac{3}{2}}\,\phi\right)$



Singularities are the limit of predictability of General Relativity.

Close to singularities, high E and $T \quad \Rightarrow \quad$ Quantum effects?

Canonical Quantum Cosmology and the Problem of Time

Promote the variables and $\mathcal{H}$ to operators in the canonical way:

$\boxed{\hat{\mathcal{H}}\,|\psi\rangle = 0}$ Wheeler-DeWitt equation: Problem of Time!

Various ways to solve it. We use *time after quantization* approach.

Example: change variable to $\alpha = \log(a)$ and use a scalar field:

$$\left(\frac{1}{6}\,\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} - \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\right)\psi(\alpha,\phi) = 0$$

Klein-Gordon equation with time ϕ. Solutions are plane waves:

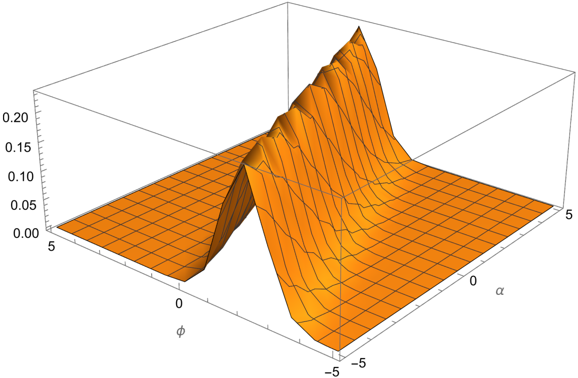
$$\psi_k(\alpha,\phi) \propto \exp\left(i\,k\left(\frac{\alpha}{\sqrt{6}} - \phi\right)\right)$$

Wavepackets and Expectation Values

We construct wavepackets with Gaussian-like weight W:

$$\Psi = \int dk\, W(k)\, \psi_k \qquad \left\langle \hat{O} \right\rangle = \int d\alpha\, i \left(\Psi^* \partial_\phi (\hat{O}\Psi) - (\hat{O}\Psi) \partial_\phi \Psi^* \right)$$

Usually $\langle \hat{\alpha} \rangle$ follows $\alpha(\phi)$ and singularities are not solved:



A different approach is needed: alternative quantization.

Outline

How it All Began: the KMM-GUP Representation

Developed to introduce a minimal length as expected in QG.

Deform the Canonical Commutation Relations:

$$[\hat{q}, \hat{p}] = i\,(1 + B\,\hat{p}^2) \qquad B > 0$$

Obtain modified uncertainty relations appearing in String Theories:

$$2\,\Delta q\,\Delta p \geq \left(1 + B\Delta p^2 + B\,\langle \hat{p} \rangle^2\right)$$

Absolute minimal uncertainty on position: $\Delta q_{\min} = \sqrt{B}$

This framework modifies the position operator:

$$\boxed{\hat{q}\,\psi(p) = i\,(1 + Bp^2)\,\partial_p \psi} \qquad \hat{p}\,\psi(p) = p\,\psi(p)$$

EV integrals need a measure $\frac{dp}{1+Bp^2}$ to preserve symmetry of $\hat{q}$.

Maggiore (PLB 1993). Kempf, Mangano, Mann (PRD 1995).

Deformed Algebras and General Properties

Generalize KMM to other functions of momentum: $[\hat{q}, \hat{p}] = i\, f(\hat{p})$

Two different (in/equivalent?) operatorial representations:

$$\boxed{\hat{q}\,\psi(p) = i\, f(p)\, \partial_p \psi} \qquad \hat{p}\,\psi(p) = p\,\psi(p) \qquad \text{(and measure for EVs)}$$

$$\hat{q}\,\psi(p) = i\, \partial_p \psi \qquad \boxed{\hat{p}\,\psi(p) = g(p)\,\psi(p)} \qquad g^{-1} = \int dp/f(p)$$

Algebra	$f(p)$	$g(p)$	Cut-off?
KMM-GUP	$1 + Bp^2$	$\frac{\tan\left(\sqrt{B}\, p\right)}{\sqrt{B}}$	$\Delta q \geq \Delta q_{\min} = \sqrt{B}$
Loop	$\sqrt{1 - \mu^2 p^2}$	$\frac{\sin(\mu p)}{\mu}$	$\lvert\langle \hat{p} \rangle\rvert \leq \frac{1}{\mu}$
Brane	$\sqrt{1 + Bp^2}$	$\frac{\sinh\left(\sqrt{B}\, p\right)}{\sqrt{B}}$	$\Delta q_{\min}$ (not always)
LUP	$1 - \mu^2 p^2$	$\frac{\tanh(\mu p)}{\mu}$	$\lvert\langle \hat{p} \rangle\rvert \leq \frac{1}{\mu}$?

Battisti (PRD 2009). **GB**, Giovannetti, Montani (IJGMMP 2022).

Semiclassical Limit and Equivalence

The usefulness of algebras is evident in their semiclassical limit.

This makes the implementation to any system very straightforward.

Fundamental	Modified-$\hat{p}$ repr.	Modified-$\hat{q}$ repr.
$[\hat{q},\hat{p}] = i\, f(\hat{p})$	$\hat{p}\,\psi = g(p)\,\psi$	$\hat{q}\,\psi = -i\, f(p)\,\partial_p\psi$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$\{q,p\} = f(p)$	$\mathcal{H}_{\text{mod}} = \mathcal{H}_{\text{class}}(q,g)$	???

First two are equivalent in isotropic Cosmology (also quantum).

The limit for the GUP-like representation has not been found.

To avoid inconsistencies, we use deformed Poisson brackets.

GB, Giovannetti, Montani (IJGMMP 2022).

Outline

Deformed FLRW Model

We start with semiclassical Loop algebra in flat isotropic FLRW:

$$\mathcal{H}_{\mathrm{PQM}}(a, p_a) = -\frac{1}{12a}\frac{p_a^2}{a} + \rho\, a^3 = 0 \qquad \{a, p_a\} = \sqrt{1 - \mu_a^2 p_a^2}$$

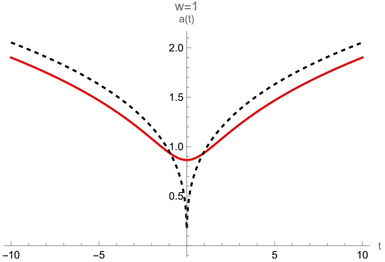
Obtain modified Friedmann equation with regularizing density:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3}\boxed{\left(1 - \frac{\rho}{\rho_{a\mu}}\right)} \qquad \rho_{a\mu} = \rho_{a\mu}(a) = \frac{1}{12\mu_a^2 a^4}$$

Critical point $\quad \rho = \rho_{a\mu} \quad \Rightarrow \quad$ What happens to the singularity?

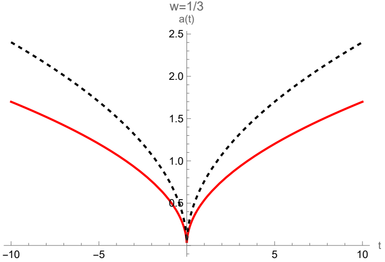
Montani, Mantero, Bombacigno, Cianfrani, **GB** (PRD 2019).

The Big Bounce



Big Bounce bridges collapse and expansion: singularity removed!

But only for specific matter with $w > 1/3$.



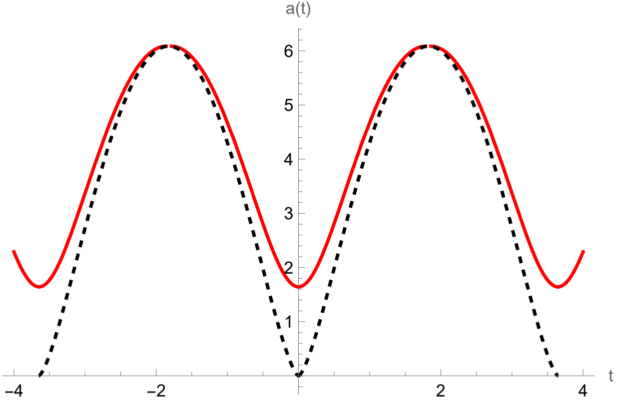
Furthermore, $\rho_{a\mu} = \rho_{a\mu}(a) \quad \Rightarrow$ QG effects at low energies?

Montani, Mantero, Bombacigno, Cianfrani, **GB** (PRD 2019).

Cyclical Universe

Classically, models with $K > 0$ have maximum and recollapse.

With Loop algebra, both critical points coexist: cyclical universe!



But maximum and minimum depend on the same initial conditions.

Either QG at low energies or the Universe is never classical.

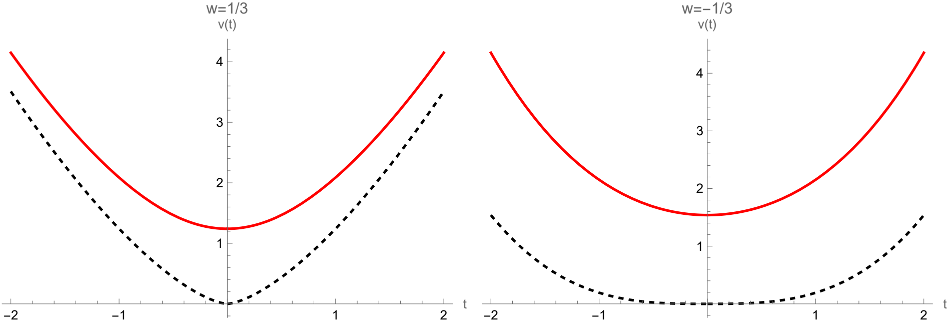
Montani, Mantero, Bombacigno, Cianfrani, **GB** (PRD 2019).

The Volume Variable

To solve these problems, change to $v = a^3$ (before deformation).

$$\mathcal{H}_{\mathrm{PQM}}(v, p_v) = -\frac{3}{4}\, p_v^2\, v + \rho\, v = 0 \qquad \{v, p_v\} = \sqrt{1 - \mu_v^2 p_v^2}$$

$$H^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{v\mu}}\right) \qquad \boxed{\rho_{v\mu} = \frac{3}{4\mu_v^2} = \text{const.}}$$



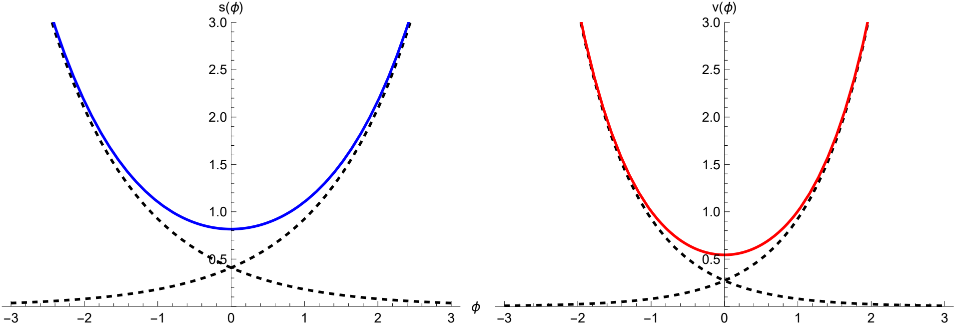
Universal Bounce! Same as effective Loop Quantum Cosmology.

Montani, Mantero, Bombacigno, Cianfrani, **GB** (PRD 2019).

Volume vs Ashtekar Variables

Compare also area $s = a^2$ and volume $v = a^3$; use ϕ as time.

$$H_s^2 = \left(\frac{\dot{s}}{2s}\right)^2 = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_{\mu s}(s)}\right) \qquad H_v^2 = \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_{\mu v}}\right)$$



Non/universality of the Bounce still dependent on the variable.

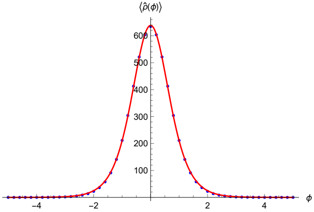
Giovannetti, **GB**, Mandini, Montani (Universe 2022).

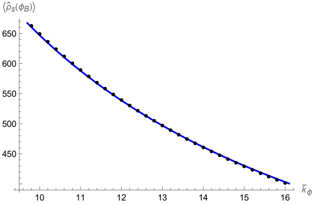
Deformed Quantum Cosmology

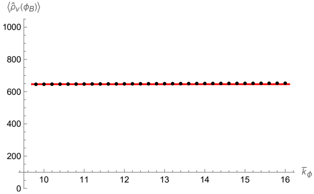
Quantize the systems (s, p_s) and (v, p_v) with Loop algebra:

$$[\hat{s}, \hat{p}_s] = i\sqrt{1 - \mu_s^2\, \hat{p}_s^2} \qquad [\hat{v}, \hat{p}_v] = i\sqrt{1 - \mu_v^2\, \hat{p}_v^2}$$

Time evolution of $\langle \hat{\rho} \rangle$ on wavepackets implies quantum Bounce.







Also on the quantum level, Bounce depends on deformed variable.

Giovannetti, **GB**, Mandini, Montani (Universe 2022).

The Problem of Equivalence

Different variables $\Rightarrow$ Different dynamics

With deformed algebras, variable is chosen <u>before</u> deformation.

What happens by changing variable <u>after</u> deformation?

1) Fix exact expression of the Loop Algebra:
similar dynamics in the two variables, but now it depends on s/v

$$\sqrt{1-\mu_v^2\, p_v^2} = \sqrt{1-\mu_1^2\, p_s^2} \quad \text{with} \quad \mu_1 = \frac{2}{3}\frac{\mu_v}{\sqrt{s}}$$

2) Fix functional form of the Loop Algebra:
new algebra depends only on p, but non-local momentum operator

$$p_s = \frac{1}{\mu_s}\sin\left(\frac{2}{3\sqrt{s}}\ \mathrm{asin}(\mu_v p_v)\right)$$

Both are problematic for quantization. Open problem.

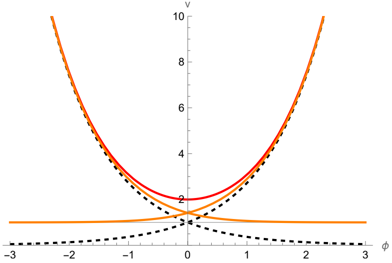
Giovannetti, **GB**, Mandini, Montani (Universe 2022).

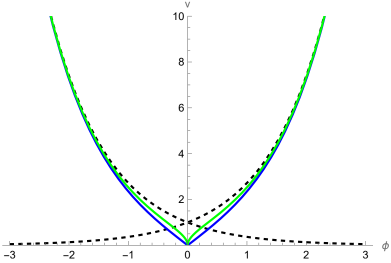
Outline

General Semiclassical Dynamics

Isotropic FLRW model with volume v and field ϕ as time.

	Square Root	Quadratic
MINUS	$\{v, p_v\} = \sqrt{1 - \mu^2 p_v^2}$	$\{v, p_v\} = 1 - \mu^2 p_v^2$
non-singular	$H^2 = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_{\mu v}}\right)$	$H^2 = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{2\rho_{\mu v}}\right)^2$
PLUS	$\{v, p_v\} = \sqrt{1 + B p_v^2}$	$\{v, p_v\} = 1 + B p_v^2$
singular	$H^2 = \frac{\rho_\phi}{3}\left(1 + \frac{\rho_\phi}{\rho_{Bv}}\right)$	$H^2 = \frac{\rho_\phi}{3}\left(1 + \frac{\rho_\phi}{2\rho_{Bv}}\right)^2$





Same as effective LQC. Same as Brane Cosmology (Strings).

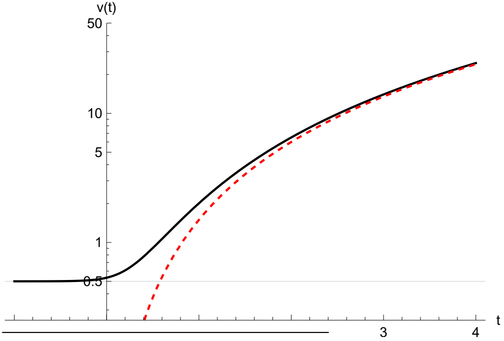
GB, Giovannetti, Montani (IJGMMP 2022).

The Emergent Universe Model

Classical FLRW with: $K > 0$ $\rho_\gamma = \dfrac{\overline{\rho_\gamma}}{v^{\frac{4}{3}}}$ $\rho_\Lambda = \text{const.}$

Fine-tuning: $\overline{\rho_\gamma}\,\rho_\Lambda = \dfrac{9}{4} K^2$ $\Rightarrow$ Einstein-static phase

We obtain it with the LUP algebra without fine-tuning:



$$\{v, p_v\} = 1 - \mu^2 p_v^2$$

$$H^2 = \frac{\rho}{3} \boxed{\left(1 - \frac{\rho}{2\rho_\mu}\right)^2}$$

$$\rho_\mu = \text{const.}$$

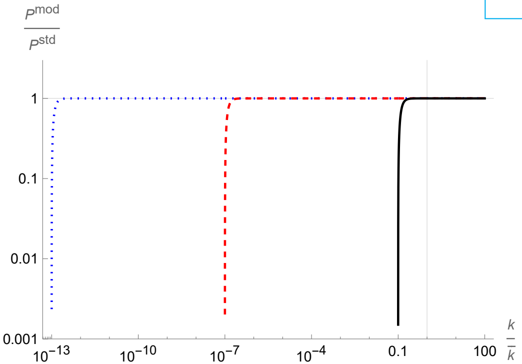
Ellis, Maartens (CQG 2003), **GB**, Montani, Melchiorri (PRD 2023).

Deformed Primordial Power Spectrum

Modify Mukhanov-Sasaki variable: $\left[\hat{\xi}_k, \hat{\pi}_k\right] = i\,(1-\mu^2\hat{\pi}_k^2)$

$$i\,\frac{\partial\psi}{\partial t} = \frac{1}{2}\left(\frac{\tanh^2(\mu\,\pi_k)}{\mu^2} - \omega_k^2(t)\,\frac{\partial^2}{\partial\pi_k^2}\right)\psi \qquad \omega_k^2(t) = k^2 - \frac{2}{t^2}$$

Solve with perturbation theory $\Rightarrow$ $\boxed{\dfrac{\mathcal{P}^{\mathrm{mod}}}{\mathcal{P}^{\mathrm{std}}} = \left(1 - \dfrac{\mu^2}{10^{65}}\,\dfrac{\bar{k}^5}{k^5}\right)}$



$\mu = 1$

$\mu = 10^{15}$

$\mu = 10^{30}$

GB, Montani, Melchiorri (PRD 2023).

Outline

Deformed Algebras in Higher Dimensions

Generalize Algebras to higher-dimensional systems (2D/3D):

$$[\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}\, f(|\hat{p}_{\rm tot}|) \qquad p_{\rm tot}^2 = \sum_i p_i^2$$

The Jacobi identities yield non-commutativity:

$$\boxed{[\hat{q}_i, \hat{q}_j] \propto A(\hat{p}_{\rm tot})\, \hat{J}_{ij}^f} \qquad A(p) \propto \frac{f\, f'}{p} \qquad J_{ij}^f \propto \frac{x_i p_j - x_j p_i}{f(p)}$$

Many complications, only the original KMM representation works:

$$\hat{q}_i\, \psi = i\, f(p_{\rm tot})\, \partial_{p_i} \psi \qquad \hat{p}_i\, \psi = p_i\, \psi$$

Fadel, Maggiore (PRD 2022).

Non-Commutative KMM-GUP on Bianchi I

Anisotropic extension of flat FLRW with Cosmological Constant.

Isotropic variable $\alpha \propto \ln(a)$ and anisotropies $\beta_\pm$

$$\mathcal{H} = \frac{e^{-3\alpha}}{12}(-p_\alpha^2 + p_+^2 + p_-^2) + \Lambda e^{3\alpha} = 0$$

Leave α unmodified, can be used as time: $[\hat{\alpha}, \hat{p}_\alpha] = i$

Deformed anisotropies form a non-commutative 2D space:

$$\left[\hat{\beta}_\pm, \hat{p}_\pm\right] = i\Big(1 + B(\hat{p}_+^2 + \hat{p}_-^2)\Big)\delta_\pm$$

$$\left[\hat{\beta}_+, \hat{\beta}_-\right] = 2\, i\, B\left(1 + B(p_+^2 + p_-^2)\right)\hat{J} \qquad \hat{J} = \frac{\hat{\beta}_+\hat{p}_- - \hat{\beta}_-\hat{p}_+}{1 + B(\hat{p}_+^2 + \hat{p}_-^2)}$$

There is an absolute minimal uncertainty in both directions.

Study wavepacket evolution in (β_+, β_-) space.

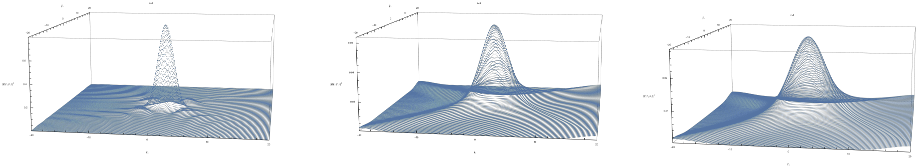
Segreto, Montani (EPJC 2024)

Non-Commutative KMM-GUP on Bianchi I

Actual uncertainty on wavepackets depends on initial position.

Minimal uncertainty obtainable only at $(\beta_+, \beta_-) = (0, 0)$.

Wavepackets evolve very slowly in α, favouring initial configuration.



Due to Vilenkin approximation and minimal uncertainties,

wavepackets can be peaked only close to the origin.

Could help explain the isotropization of the Universe?

Segreto, Montani (EPJC 2024)

Outline

Summary and Outlook

- Modified Algebras are a versatile formalism and an easy way to introduce cut-off effects in any system
- Some forms reproduce effects from other QG theories and yield e.g. Loop Quantum Cosmology and Brane Cosmology
- Future project: develop reconstruction method to rewrite other QG theories as deformed algebras?
- The deformed dynamics strongly depend on the chosen variable, canonical transformations must be found
- Next step: more complicated settings such as other Bianchi models and combining different algebras

Work in Progress: Bouncing Bianchi II

What if we combine different algebras for different variables?

Bianchi models with Misner-like variables $v = e^{3\alpha}$, $\beta_\pm$

We deform the isotropic sector with Loop algebra:

$$[\hat{v}, \hat{p}_v] = i\sqrt{1 - \mu_v^2 \hat{p}_v^2}$$ Singularity replaced by Bounce

Deformed anisotropies form a non-commutative 2D space:

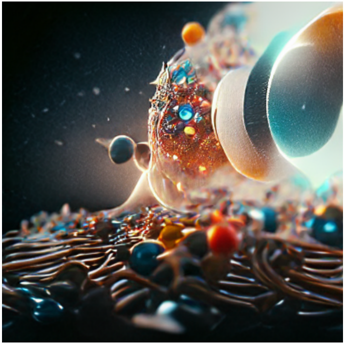
$$\left[\hat{\beta}_\pm, \hat{p}_\pm\right] = i\, f(\hat{p}_+^2 + \hat{p}_-^2) \qquad \left[\hat{\beta}_+, \hat{\beta}_-\right] = i\,\frac{A(\hat{p}_{\text{tot}})}{f(\hat{p}_{\text{tot}})}\left(\hat{\beta}_+\hat{p}_- - \hat{\beta}_-\hat{p}_+\right)$$

- Quantum operators are complicated: work in classical limit.
- $v(t)$ is not monotonic anymore. What time can we use?

We expect a deformed BKL map. Stay tuned...

GB, Gielen (in preparation, 2024).

Thanks for your attention!



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